THE SOLUTIONS

OF

GEOMETRICAL PROBLEMS,

CONSISTING CHIEFLY OF

EXAMPLES IN PLANE CO-ORDINATE GEOMETRY,

PROPOSED AT

ST JOHN'S COLLEGE, CAMBRIDGE,

FROM DEC 1830 TO DEC. 1846.

WITH

AN APPENDIX,

CONTAINING SEVERAL GENERAL PROPERTIES OF CURVES OF THE SECOND ORDER, AND THE DETERMINATION OF THE MAGNITUDE AND POSITION OF THE AXES OF THE CONIC SECTION REPRESENTED BY THE GENERAL EQUATION OF THE SECOND DEGREE.

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TO

WILLIAM CRACKANTHORPE, Esq.,

THIS BOOK

IS INSCRIBED AS A TESTIMONY OF THE HIGHEST RESPECT

AND ESTEEM,

AND AS A GRATEFUL ACKNOWLEDGMENT OF

NUMEROUS AND EXTENSIVE OBLIGATIONS

CONFERRED UPON

THE AUTHOR.

PREFACE.

The Examination Papers which the Author has selected for solution in the present Treatise have been proposed in the several years from 1830 to 1846 to the students of St John's College at the end of their fourth term of residence, and according to the plan which he adopted in the solution of the Trigonometrical Problems, he has endeavoured to place them before the reader in a proper form for the inspection of the examiner. The problems are sufficiently varied in their character to exercise the student in the ordinary properties of the straight line, circle, and conic sections; they have been proposed by some of the most distinguished members of the society; the generality of the results are remarkable for their neatness and simplicity; and except in one instance it is needless here to make any further comment.

In Question 6, Dec. 1833. (No. IV), it is required "To inscribe in a circle a triangle whose sides or sides produced shall pass through three given points in the same plane."

This problem has been solved analytically in a most ingenious and elegant treatise, entitled "Researches on Curves of the second order" lately published by Mr Hearn,

who remarks that it was proposed by M. Cramer to M. de Castillon, and that Lagrange has given a purely analytical solution which may be found in the memoirs of the Academy of Berlin (1776). The problem then being one of acknowledged difficulty, the author hopes that the first Appendix in which an analytical solution has been given when a *conic section* is substituted for the circle, will not be entirely devoid of interest.

The case of the three different conic sections has been separately considered, and the author has afterwards still further generalized the problem, by inscribing in a given conic section, a polygon whose n sides taken in order shall pass through n fixed points.

A simple and concise geometrical solution of M. Cramer's problem has been extracted from "The Liverpool Apollonius by J. H. Swale," and inserted in the third Appendix; and when the triangle is to be inscribed in a conic section, the problem has been reduced to that of inscribing in a circle a triangle whose three sides shall pass through three fixed points, so as to afford a comparatively simple geometrical solution in the more general case.

Some apology may be considered necessary for the introduction into the second appendix of two different methods of determining the magnitude and position of the conic section represented by the general equation of the second degree.

By transferring the equation to the focus the author has endeavoured to point out the change which takes place when the curve approaches to a parabola; and on that account he has deduced the latus rectum and the co-ordinates of the vertex of the parabola from those of the ellipse in its transition state.

The reduction of the equation to the focus led to the forms which have been determined for the elements of the curve; and it was afterwards found that most of them might be obtained more briefly by the polar equation from the centre.

In the case of oblique co-ordinates the expressions are remarkably symmetrical, and are placed in such a form that the reader may perceive at once their connexion with the corresponding results when the co-ordinates are rectangular.

In the latter part of the second appendix the author has endeavoured to trace a conic section geometrically as far as it appeared practicable, by the determination of a successive series of points when five points of the curve can in any manner be found.

A remarkably simple construction for a tangent at one of the five given points has been obtained (Art. 89); a tangent has also been drawn from a given point without the curve; and the position of a point in the conic section in any proposed direction has been determined by its points of intersection with a given straight line.

In a subject which for ages has exercised the skill and ingenuity of the most profound mathematicians, little can be expected which is really original; but as only a very small portion of the matter in the appendices has been met with by the author elsewhere, even if he may have been anticipated in any or all the properties which are inserted, it is extremely probable that the proofs now given will be widely different from any which have been hitherto published.

A reference has been made in several places to an edition of Euclid lately published by Mr Potts, in which will be found much valuable information; it is well deserving the attention of every one who wishes to study Geometry.

T. G.

CAMBRIDGE,
Nov. 1847.

GEOMETRICAL PROBLEMS.

ST JOHN'S COLLEGE. Dec. 1830. (No. I.)

1. Parallelograms upon the same base and between the same parallels are equal to one another.

2. Of unequal magnitudes, the greater has a greater ratio to the same than the less.

3. If the diameter of a circle be one of the equal sides of an isosceles triangle, the base will be bisected by the circumference.

4. The line joining the centres of the inscribed and circumscribed circles of a triangle subtends at any one of the angular points an angle equal to the semidifference of the other two angles.

5. Find a point without a given circle, such that the sum of the two lines drawn from it touching the circle, shall be equal to the line drawn from it through the centre to meet the circumference.

6. If a circle roll within another of twice its size, any point in its circumference will trace out a diameter of the first.

7. If from any point in the circumference of a circle, a chord and tangent be drawn, the perpendiculars dropped upon them from the middle point of the subtended arc, are equal to one another.

8. If α, β, γ represent the distances of the angles of a triangle from the centre of the inscribed circle, and a, b, c the sides respectively opposite to them, then

$$\alpha^2 a + \beta^2 b + \gamma^2 c = abc.$$

9. Describe a circle through a given point and touching a given straight line, so that the chord joining the given point

and point of contact, may cut off a segment capable of a given angle.

10. Shew that the perimeter of the triangle formed by joining the feet of the perpendiculars dropped from the angles upon the opposite sides of a triangle, is less than the perimeter of any other triangle whose angular points are on the sides of the first.

11. Explain what is meant by the equation to a curve; find the equation to a straight line, and state clearly the meaning of the constants involved.

12. Trace the circle whose equation is

$$a(x^2 + y^2) + b^2(x + y) = 0;$$

draw the lines represented by the equations

$$y^2 - 2xy \sec \alpha + x^2 = 0,$$

and shew that the angle between them is α.

13. The portion of a straight line intercepted by two rectangular axes, and the perpendicular upon it from their intersection are each of given length; what is the equation to the line?

14. Find the equation to an ellipse, and deduce that to the parabola from it.

15. Find the co-ordinates of the point from which if three lines be drawn to the angles of a triangle, its area is trisected.

16. In the last question, the (distance)^2 from the angle A of the required point $= \frac{2}{9}\left(b^2 + c^2 - \frac{a^2}{2}\right)$.

17. If the centre of the inscribed circle of a triangle be fixed, and α, β, γ represent the distances of its angles from any fixed point in *space;* then whatever position the triangle assume, the expression $\alpha^2 a + \beta^2 b + \gamma^2 c$ is invariable.

SOLUTIONS TO (No. I.)

1. Euclid, Prop. 35, Book I.

2. Euclid, Prop. 8. Book V.

3. Let AB, AC (fig. 1) be the equal sides of an isosceles triangle; upon AB describe a semicircle cutting the base BC in D; join AD: then $\angle ADB$ is a right angle $= \angle ADC$; also $\angle ABD = \angle ACD$, and AD is common to the two triangles ABD, ADC, $\therefore BD = DC$.

4. Let ABC (fig. 2) be a triangle, d, D the centres of the inscribed and circumscribing circles; draw DE, de perpendicular to AB, and join AD, Ad: then

$$\angle ADE = \tfrac{1}{2} \angle ADB = \angle C;$$

$$\text{or } \angle DAE = \frac{\pi}{2} - C = \frac{A+B+C}{2} - C = \frac{A+B-C}{2};$$

$$\text{also } \angle dAe = \frac{A}{2}:$$

$$\therefore \angle DAd = \angle DAE - \angle dAe = \frac{B-C}{2}.$$

In like manner it may be proved by joining DB, dB, DC, dC that

$$\angle DBd = \frac{A-C}{2}, \text{ and } \angle DCd = \frac{A-B}{2}.$$

If $\angle B$ be less than $\angle C$, $\angle DAd$ will be negative, which shews that AD would in that case lie below Ad; and so of the rest.

5. Let D (fig. 3) be the required point in any diameter ACB produced; DE a tangent drawn from D; then

$$DE^2 = DA \,.\, DB, \text{ and } DA = 2DE,$$

$$\therefore DA^2 = 4DE^2 = 4DA \,.\, DB \text{ or } DA = 4DB;$$

$$\text{hence } AB = 3BD, \text{ and } BD = \frac{AB}{3},$$

which determines the position of the point D.

6. Let C (fig. 4) be the centre of the larger circle; A the original point of contact of the two circles; let the inner circle roll until P becomes the point of contact; join CP and bisect it in O; then CP is the diameter, and O the centre of the inner circle PQC when P becomes the point of contact. Let the circle PQC cut AC in Q, and join OQ; then

$$\angle POQ = 2\angle PCQ, \text{ or } \frac{PQ}{PO} = \frac{2PA}{PC};$$

$$\therefore \text{ arc } PQ = \frac{2PO}{PC} \text{ arc } PA = \text{arc } PA;$$

$\therefore$ Q is the point which originally coincided with A, and it is in the radius AC; hence the locus of the point Q is the diameter which passes through A.

7. Let AB (fig. 5) be a chord and AT a tangent at the point A of a circle; C the middle point of the arc AB; join AC, CB and draw CM, CN perpendicular to AB, AT respectively; then

$$CN = CA \sin\angle CAT = CA \,.\, \sin\angle CBA = CB \,.\, \sin\angle CBA = CM.$$

8. Let A, B, C be the angles of the triangle respectively opposite to the sides a, b, c, and r the radius of the inscribed circle; then

$$\alpha = \frac{r}{\sin\frac{A}{2}}, \quad \beta = \frac{r}{\sin\frac{B}{2}}, \quad \gamma = \frac{r}{\sin\frac{C}{2}};$$

hence $$\alpha^2 a + \beta^2 b + \gamma^2 c = r^2 \left\{\frac{a}{\left(\sin\frac{A}{2}\right)^2} + \frac{b}{\left(\sin\frac{B}{2}\right)^2} + \frac{c}{\left(\sin\frac{C}{2}\right)^2}\right\}$$

$$= 2r^2\left(\frac{a}{\sin A}\cot\frac{A}{2} + \frac{b}{\sin B}\cot\frac{B}{2} + \frac{c}{\sin C}\cot\frac{C}{2}\right)$$

$$= \frac{2ar^2}{\sin A}\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) \text{ since } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

but $r\cot\frac{A}{2} = S - a, \quad r\cot\frac{B}{2} = S - b, \quad r\cot\frac{C}{2} = S - c,$

$$\text{where } S = \frac{a+b+c}{2};$$

$$\therefore\ r\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) = S;$$

$$\text{or } \alpha^2 a + \beta^2 b + \gamma^2 c = \frac{2a}{\sin A} rS = \frac{2a}{\sin A} \times \text{area of } \triangle ABC$$

$$= \frac{2a}{\sin A}\left(\frac{bc}{2}\sin A\right) = abc.$$

9. Let P (fig. 6) be the given point, and AB the given straight line; draw PC parallel to AB, make $\angle CPD =$ the given angle, and let PD meet AB in D; bisect DP in E, and draw DF, EF respectively perpendicular to AB, DP meeting each other in F; with centre F and distance FD describe a circle; this will pass through the point P and touch the straight line AB; and $\angle PDB =$ the angle in the alternate segment cut off by $DP = \angle CPD =$ the given angle.

10. Let b, c (fig. 7) be two of the angular points of the triangle which is required to be of the least possible perimeter; a the remaining angular point in the side BC; then $ba + ca$ will be least when $\angle Cab = \angle Bac$. (POTTS' EUCLID, p. 293.)

Now if a be the angular point properly determined, and b the angular point in the side AC, then $ac + cb$ is least when $\angle Bca = \angle Acb$: similarly, if a, c be two angular points of the triangle, $ab + bc$ will be least when $\angle Abc = \angle Cba$: hence the perimeter of the triangle abc will be the least possible when $\angle Cab = \angle Bac$; $\angle Cba = \angle Abc$, and $\angle Acb = \angle Bca$. Now if Aa, Bb, Cc be drawn from the angles A, B, C perpendicular to the opposite sides respectively, the circle described on the diameter AC will pass through the points c, a, because the angles AcC, AaC are right angles; hence

$$\angle Aac = \angle ACc = \frac{\pi}{2} - A, \text{ and } Bac = \frac{\pi}{2} - Aac = A.$$

Similarly, the circle described on the diameter AB will pass through the points b, a; and $\angle Aab = \angle ABb = \frac{\pi}{2} - A$;

$$\therefore \angle Cab = \frac{\pi}{2} - Aab = A, \text{ or } \angle Bac = \angle Cab;$$

similarly, $\angle Cba = \angle Abc$, and $\angle Acb = \angle Bca$; therefore the triangle abc has the least possible perimeter.

11. See HYMERS' CONIC SECTIONS. Art. 13.

12. (1) $x^2 + \frac{b^2 x}{a} + y^2 + \frac{b^2 y}{a} = 0$;

$$\therefore \left(x + \frac{b^2}{2a}\right)^2 + \left(y + \frac{b^2}{2a}\right)^2 = \frac{b^4}{2a^2},$$

the equation to a circle the co-ordinates of whose centre are $-\frac{b^2}{2a}$, $-\frac{b^2}{2a}$, and whose radius $= \frac{b^2}{\sqrt{2}\,a}$. Hence if Ax, Ay (fig. 8) be the co-ordinate axes, take AB, BC each $= -\frac{b^2}{2a}$, and with centre C and radius $= \frac{b^2}{\sqrt{2}\,a}$ describe a circle AED; it will pass through A and meet the axis of x in the point D such that $DB = BA$;

$$\text{and } \angle ACB = \angle DCB = \frac{\pi}{4}; \ \therefore \angle ACD = \frac{\pi}{2};$$

hence AED is a quadrant of the circle having the chord produced for the axis of x.

(2) Since $y^2 - 2xy \sec\alpha + x^2 = 0$, we have

$$\left(\frac{y}{x}\right)^2 - 2\sec\alpha\,\frac{y}{x} + (\sec\alpha)^2 = (\tan\alpha)^2;$$

$$\therefore \frac{y}{x} = \sec\alpha \pm \tan\alpha = \tan\left(45 \pm \frac{\alpha}{2}\right),$$

which are the equations to two straight lines passing through the origin and inclined at angles $45 + \frac{a}{2}$ and $45 - \frac{a}{2}$ respectively to the axis of x; hence the angle between them

$$= \left(45 + \frac{a}{2}\right) - \left(45 - \frac{a}{2}\right) = a.$$

13. Let O (fig. 9) be the origin; AB the straight line meeting the axes of x and y in the points A, B respectively; draw OP perpendicular to AB and let $AB = a$, $OP = b$, $OA = m$, $OB = n$; then equation to the line AB is

$$\frac{x}{m} + \frac{y}{n} = 1$$

where $m^2 + n^2 = a^2$, and $mn = 2 \triangle AOB = AB \,.\, OP = ab$;

$$\therefore\ m + n = \pm\sqrt{a^2 + 2ab},\ m - n = \pm\sqrt{a^2 - 2ab};$$

$$\text{or } 2m = \pm(\sqrt{a^2 + 2ab} \pm \sqrt{a^2 - 2ab}),$$

$$\text{and } 2n = \pm(\sqrt{a^2 + 2ab} \mp \sqrt{a^2 - 2ab});$$

therefore if

$\sqrt{a^2+2ab} + \sqrt{a^2-2ab} = 2\alpha$, and $\sqrt{a^2+2ab} - \sqrt{a^2-2ab} = 2\beta$, we have $m = \pm\alpha$, or $\pm\beta$, and the corresponding values of n are $\pm\beta$, $\pm\alpha$; hence $\pm\frac{x}{\alpha} \pm \frac{y}{\beta} = 1$, or $\pm\frac{x}{\beta} \pm \frac{x}{\alpha} = 1$; each of which equations will give the equations to four straight lines by the four combinations of the double sign $\pm$. If OA, OA', OA'', OA''' be taken each equal to α, and OB, OB', OB'', OB''' each equal to β; then AB, BA'', $A''B''$, $B''A$; $A'B'$, $B'A'''$, $A'''B'''$, $B'''A'$ will be the straight lines required.

14. See HYMERS' CONIC SECTIONS. Art. 104.

The equation to the ellipse is

$$y^2 = (1 - e^2)\left(\frac{2p}{1-e}x - x^2\right) = 2p(1+e)x - (1-e^2)x^2;$$

and when $e = 1$, $y^2 = 4px$, which is the equation to the parabola.

15. Let P (fig. 10) be the required point within the triangle ABC; join AP, BP, CP and produce them to meet the opposite sides in a, b, c respectively: then

$$\triangle APB = \tfrac{1}{2} AB \,.\, Pc \,.\, \sin PcB = \tfrac{1}{3} \triangle ACB$$
$$= \tfrac{1}{3} (\tfrac{1}{2} AB \,.\, Cc \,.\, \sin PcB),$$

$$\therefore\; Pc = \frac{Cc}{3}; \text{ similarly, } Pb = \frac{Bb}{3} \text{ and } Pa = \frac{Aa}{3}.$$

Join ac, then $PC = 2Pc$, $PA = 2Pa$,

and $Pa : Pc :: PA : PC$,

or ac is parallel to AC, and

$$Bc : BA :: ca : CA :: Pc : PC :: 1 : 2;$$

$\therefore$ $BA = 2Bc$, and the side BA is bisected in C; and in like manner it may be proved that a, b are the middle points of BC, CA respectively. Draw PM parallel to AC, then if AB, AC be taken for the co-ordinate axes, and x, y be the co-ordinates of P;

$$PM : AC :: Pc : Cc, \text{ or } PM = \tfrac{1}{3} AC;$$

$$\therefore\; y = \frac{b}{3}, \text{ and } AM : MP :: Ac : ca :: \frac{c}{2} : \frac{b}{2};$$

$$\therefore\; x : y :: \frac{c}{2} : \frac{b}{2} \text{ and } x = \frac{cy}{b} = \frac{c}{3}.$$

16. Let $PA = a$, then $Aa = \dfrac{3a}{2}$: and since BC is bisected in a,

$$2Aa^2 + 2Ba^2 = AB^2 + AC^2, \text{ or } \frac{9a^2}{2} + \frac{a^2}{2} = b^2 + c^2;$$

$$\therefore\; a^2 = \frac{2}{9}\left(b^2 + c^2 - \frac{a^2}{2}\right).$$

17. First let P (fig. 11) be a point in the plane ABC, O the centre of the inscribed circle whose radius is r; join PA, PB, PC, PO, AO, BO, CO; then if $PA = \alpha$, $PB = \beta$, $PC = \gamma$, $\angle AOP = \theta$, we have

$$\alpha^2 a + \beta^2 b + \gamma^2 c = a\,(PO^2 + AO^2 - 2PO\,.\,AO\,.\cos POA)$$
$$+ b\,(PO^2 + BO^2 - 2PO\,.\,BO\,.\cos POB)$$
$$+ c\,(PO^2 + CO^2 - 2PO\,.\,CO\,.\cos POC)$$
$$= (a + b + c)\,PO^2 + a\,.\,AO^2 + b\,.\,BO^2 + c\,.\,CO^2$$
$$-2PO(a.AO.\cos POA + b\,.\,BO\,.\cos POB + c\,.\,CO\,.\cos POC)$$
$$= (a + b + c)\,PO^2 + abc$$
$$-2PO\left\{a.\frac{r}{\sin\frac{A}{2}}\cos\theta + b\frac{r}{\sin\frac{B}{2}}\cos\left(\theta+\pi-\frac{A+B}{2}\right)+c\frac{r}{\sin\frac{C}{2}}\cos(\pi-\frac{A+C}{2}-\theta)\right\}$$

(from Quest. 8)

$$= (a + b + c)\,PO^2 + abc$$
$$-4PO.\frac{ar}{\sin A}\left\{\cos\frac{A}{2}\cos\theta - \cos\frac{B}{2}\sin\left(\frac{C}{2}+\theta\right) + \cos\frac{C}{2}\sin\left(\theta-\frac{B}{2}\right)\right\}$$
$$= (a + b + c)\,PO^2 + abc$$
$$-4PO\,.\,\frac{ar}{\sin A}\left\{\cos\frac{A}{2}\cos\theta - \left(\sin\frac{C}{2}\cos\frac{B}{2} + \cos\frac{C}{2}\sin\frac{B}{2}\right)\cos\theta\right\}$$
$$= (a + b + c)\,PO^2 + abc - 4PO\,\frac{ar}{\sin A}\left(\cos\frac{A}{2} - \sin\frac{B+C}{2}\right)\cos\theta$$
$$= (a + b + c)\,PO^2 + abc,$$

$$\text{since } \sin\frac{B+C}{2} = \cos\frac{A}{2}.$$

Next let P be without the plane ABC; draw PQ perpendicular to the plane, and join QA, QB, QC, QO, then

$$PA^2 = QP^2 + QA^2,\ \ PO^2 = QP^2 + QO^2;$$

and $\alpha^2 a+\beta^2 b+\gamma^2 c = a(QP^2+QA^2)+b(QP^2+QB^2)+c(QP^2+QC^2)$

$$= (a + b + c)\,QP^2 + a.QA^2 + b.QB^2 + c.QC^2$$
$$= (a + b + c)\,QP^2 + (a + b + c)\,QO^2 + abc$$

(by the first case) $= (a + b + c)\,PO^2 + abc$, and is therefore invariable, since P and O are two fixed points.

ST JOHN'S COLLEGE. Dec. 1831. (No. II.)

1. If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

2. If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two; the first two and the other two shall contain equal angles.

3. The sum of the perpendiculars from any point in the base of an isosceles triangle is equal to a line of fixed length.

4. To find a point in the side or side produced of any parallelogram, such that the angle it makes with the line joining the point and one extremity of the opposite side, may be bisected by the line joining it with the other extremity.

5. The lines which bisect the vertical angles of all triangles on the same base and with the same vertical angle, all intersect in one point.

6. If a semicircle be described on the hypothenuse AB of a right-angled triangle ABC, and from the centre E the radius ED be drawn at right angles to AB, shew that the difference of the segments on the two sides equal twice the sector CED.

7. The locus of the centres of the circles which are inscribed in all right-angled triangles on the same hypothenuse is the quadrant described on the hypothenuse.

8. Of all the angles which a straight line makes with any straight lines drawn in a given plane to meet it, the least is that which measures the inclination of the line to the plane.

9. Find the sine of the inclination to each other of two straight lines whose equations are given.

10. Find the length of the perpendicular from the origin of co-ordinates upon the line whose equation is

$$a(x-a)+b(y-b)=0,$$

and the part of the line intercepted between the co-ordinate axes.

11. The equation to a circle is $y^2+x^2=a(y+x)$; what is the equation to that diameter which passes through the origin of co-ordinates?

12. A side of a triangle being assumed as the axis of x, the equations to the equations to the other sides are $y=ax+b$, and $y=a'x$; determine the sides and angles of the triangle.

13. If through any point of a quadrant whose radius is R, two circles be drawn touching the bounding radii of the quadrant, and r, r' be the radii of these circles, shew that $rr'=R^2$.

SOLUTIONS TO (No. II.)

1. Euclid, Prop. 25. Book v.

2. Euclid, Prop. 10. Book xi.

3. Let D (fig. 12) be a point in the base AB of the isosceles triangle ABC; draw DE, DF perpendicular to AC, BC respectively; then

$$DE + DF = AD \cdot \sin A + DB \cdot \sin B = (AD + DB) \sin A$$
$$= AB \cdot \sin A$$

equal the perpendicular from B upon the side AC, and is therefore constant wherever D be taken in AC.

4. Let $ABCD$ (fig. 13) be the parallelogram; with centre B and radius BA describe a circle cutting DC in the points E, F; join AF, BF; then $BF = BA$ and

$$\angle BFA = \angle BAF = \angle AFD,$$

or $\angle DFB$ is bisected by the straight line AF. Similarly if AE, BE be joined, $\angle DEB$ is bisected by AE. Also if with centre A and radius AB a circle be described cutting DC in the points G, H, the angles CGA, CHA will be bisected by the straight lines BG, BH respectively.

5. Let ACB (fig. 14) be one of the triangles; about it describe the circle $ACBD$; then since the vertical angle is constant, the vertices of all the triangles will be in the circumference ACB. Bisect the arc AB in D, and join CD; then since arc AD = arc DB, $\angle ACD = \angle BCD$, and CD bisects the angle ACB; hence the line which bisects the angle ACB will always pass through the fixed point D.

6. Let ABC (fig. 15) be the right-angled triangle, $ADCB$ the semicircle described upon the hypothenuse AB; then since $AE = EB$, $\triangle AEC = \triangle BEC$; and segment on AC − segment on BC = (segment on $AC + \triangle AEC$) − (segment on $BC + \triangle BEC$) = sector AEC − sector BEC = (sector AED + sector DEC) − (sector BED − sector DEC) = 2 sector DEC since quadrant AED = quadrant BED.

7. Let O (fig. 16) be the centre of the circle inscribed in the $\triangle ACB$; join OA, OB;

$$\text{then } \angle AOB = \pi - (OAB + OBA)$$

$$= \pi - \frac{A+B}{2} = \pi - \frac{\pi - C}{2} = \frac{\pi}{2} + \frac{C}{2};$$

and is constant. Hence the locus of the point O is a segment of a circle containing an angle

$$= \frac{\pi}{2} + \frac{C}{2};$$

and the $\angle ADB$ in the alternate segment

$$= \pi - \left(\frac{\pi}{2} + \frac{C}{2}\right) = \frac{\pi}{2} - \frac{C}{2};$$

therefore the angle subtended by AOB at the centre $E = \pi - C$.

When the $\triangle ACB$ is right-angled,

$$\angle C = \frac{\pi}{2}; \text{ and } \angle AEB = \pi - C = \frac{\pi}{2};$$

therefore AOB is the quadrant described upon the hypothenuse AB.

8. Let P (fig. 17) be any point in the line AP which meets the plane in the point A; draw PM perpendicular to the plane; join AM, and through A draw any other line AQ in the plane; draw PQ perpendicular to AQ, and join QM; then if θ, θ', be the inclinations of AP to AM, AQ respectively,

$$\sin\theta = \frac{PM}{AP}, \text{ and } \sin\theta' = \frac{PQ}{AP};$$

but $PQ^2 = PM^2 + MQ^2$ since $\angle PMQ$ is a right angle, or PQ is greater than PM; therefore $\sin\theta$ is less than $\sin\theta'$ and θ is less than θ'. Now θ measures the inclination of the line AP to the plane, which is therefore less than the inclination of the line AP to any other line AQ drawn to meet it in that plane.

9. Let $y = ax + b$, $y = a'x + b'$ be the equations to the two straight lines; then if θ, θ' be the angles which they respectively make with the axis of x,

$$\tan\theta = a, \quad \tan\theta' = a',$$

and $\sin(\theta' - \theta) = \cos\theta\cos\theta'(\tan\theta' - \tan\theta) = \dfrac{a' - a}{\sqrt{(1+a^2)(1+a'^2)}}$

which is the sine of the inclination of the two straight lines to each other.

10. Let O (fig. 9) be the origin, and A, B the points in which the given straight line meets the axes of x and y respectively; then the equation to the line OP drawn through O perpendicular to AB is $y = \dfrac{bx}{a}$; and if x', y' be the co-ordinates of the point of intersection P, we have

$$a(x' - a) + b\left(\frac{bx'}{a} - b\right) = 0,$$

$$\text{or } x' = a, \text{ and } y' = \frac{bx'}{a} = b;$$

therefore the length of the perpendicular

$$OP = \sqrt{x'^2 + y'^2} = \sqrt{a^2 + b^2}.$$

Again in the equation to the given straight line, make $y = 0$, the corresponding value of x is $OA = \dfrac{a^2 + b^2}{a}$; and by making $x = 0$, the value of y is $BO = \dfrac{a^2 + b^2}{b}$;

$$\therefore AB = \sqrt{AO^2 + BO^2} = \frac{(a^2 + b^2)^{\frac{3}{2}}}{ab}.$$

11. $x^2 - ax + y^2 - ay = 0,$

$$\therefore \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{2}$$

the equation to a circle, the co-ordinates of whose centre are

$$\frac{a}{2}, \frac{a}{2}.$$

Let $y = mx$ be the equation to the diameter which passes through the origin, then since this straight line passes through a point $\dfrac{a}{2}$, $\dfrac{a}{2}$, we have $\dfrac{a}{2} = m \cdot \dfrac{a}{2}$ or $m = 1$; and $y = x$ is the equation required.

12. Let ABC (fig. 18) be the triangle, A the origin, and BA produced the axis of x, the equations to AC, BC are $y = a'x$, and $y = ax + b$;

$$\therefore \tan (CAx) = \tan (\pi - A) = a' \text{ or } \tan A = -a',$$

$$\text{and } \tan CBx = \tan B = a;$$

$$\text{hence } \tan C = \tan (CAx - CBx) = \frac{a' - a}{1 + aa'}.$$

Draw CM perpendicular to AB, and in the equation to BC make $y = 0$, then the corresponding value of $x = -AB$ $= -\frac{b}{a}$ or $AB = \frac{b}{a}$: let x', y' be the co-ordinates of M, then

$$a'x' = ax' + b \text{ or } x' = -AM = \frac{b}{a' - a};$$

$$y' = CM = a'x' = \frac{a'b}{a' - a}; \quad \therefore AC = \sqrt{x'^2 + y'^2} = \frac{b\sqrt{1 + a'^2}}{a' - a};$$

$$\text{also } BM = AB - AM = \frac{b}{a} + \frac{b}{a' - a} = \frac{ba'}{a(a' - a)};$$

$$\therefore CB = \frac{a'b\sqrt{1 + a^2}}{a(a' - a)}.$$

13. Let the bounding radii AB, AC (fig. 19) of the quadrant be taken for the axes of x and y respectively; bisect the $\angle BAC$ by the straight line AD, and from any point D in AD draw DE perpendicular to AB, then a circle described with centre D and radius DE will touch both the bounding radii AB, AC: and if $AE = ED = \rho$, the radius will also be $= \rho$, and the equation to the circle is

$$(x - \rho)^2 + (y - \rho)^2 = \rho^2, \text{ or } x^2 + y^2 - 2\rho(x + y) + \rho^2 = 0.$$

Let this circle cut the quadrant in the point P whose co-ordinates are h, k;

$$\therefore h^2 + k^2 - 2\rho(h + k) + \rho^2 = 0, \text{ or } R^2 - 2(h + k)\rho + \rho^2 = 0;$$

from this equation the two values r, r' of ρ may be determined, and $rr' = R^2$, $r + r' = 2(h + k)$.

ST JOHN'S COLLEGE. Jan. 1833. (No. III.)

1. Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

2. If a solid angle be contained by three plane angles, any two of them are greater than the third.

3. If a straight line be divided into any two parts; to find a point without the line at which the segments shall contain equal angles.

4. Given the base and sum of the sides containing the vertical angle, and the line drawn from one extremity of the base perpendicular to it to meet a side or side produced; to construct the triangle.

5. In a triangle ABC, if CD be drawn bisecting the vertical angle C, and meeting the base in D; and DE, DF be drawn respectively parallel to the sides AC, BC and meeting them in E and F; prove that $DE = DF$.

6. If two circles be described about the same centre, the radius of one being double that of the other, and a point be taken within the inner circle; to draw from this point to the outer circumference a straight line which shall be bisected by the inner circumference.

7. ACB is a segment of a circle, and any chord AC is produced to a point P, so that $AC : CP$ in a given ratio. Required to find the locus of P.

8. If $FACB$ be a line passing through the centre C of a circle, CD a radius perpendicular to the diameter ACB; DEF, DGH any two lines cutting the circle in E, H, and the straight line $FACB$ in F, G; shew that a circle may be made to pass through the four points F, E, G, H.

9. If through any point O within a triangle, three straight lines be drawn from the angles A, B, C to meet the opposite sides in a, b, c respectively; prove that

$$Ac \,.\, Ba \,.\, Cb = cB \,.\, aC \,.\, bA.$$

10. Find the equation to a straight line which cuts a given circle, when the lines drawn from the points of intersection to the centre contain a right angle, and one of them is inclined to the axis of x at a given angle.

11. If from a point without a given circle, two straight lines be drawn touching the circle; find the equation to the line joining the points of contact.

12. If a_1, a_2 be the sides of a right-angled triangle, and δ_1, δ_2 the diameters of the circles inscribed in the triangles formed by joining the vertex and the middle point of the hypothenuse; shew that

$$\frac{1}{\delta_1} - \frac{1}{\delta_2} = \frac{1}{a_2} - \frac{1}{a_1}.$$

13. Find the equation to the straight line drawn from a given point to bisect a given equilateral triangle.

SOLUTIONS TO (No. III).

1. Euclid, Prop. 9. Book v.

2. Euclid, Prop. 20, Book xi.

3. Let AB (fig. 20) be divided into two parts in the point C; E the required point such that $\angle AEC = \angle BEC$; then if $AC = c$, $BC = c'$, $CE = \rho$; $\angle ECB = \phi$,

$$\angle AEC = \angle BEC = \theta, \ \frac{\rho}{c'} = \frac{\sin(\phi + \theta)}{\sin\theta}, \ \frac{\rho}{c} = \frac{\sin(\phi - \theta)}{\sin\theta};$$

$$\therefore \left(\frac{1}{c'} - \frac{1}{c}\right)\rho = \frac{\sin(\phi + \theta) - \sin(\phi - \theta)}{\sin\theta} = 2\cos\phi,$$

or $\rho = \dfrac{2cc'}{c - c'}\cos\phi$, which is the polar equation to a circle whose diameter $= \dfrac{2cc'}{c-c'}$, is in the direction CB.

Produce CB to D, take $CD = \dfrac{2cc'}{c-c'}$, and upon CD describe a circle; from any point E in its circumference draw AE, EC, EB, then will $\angle AEC = \angle BEC$.

4. Let AB (fig. 21) be the given base; draw AC perpendicular to the base, and of the given length, then BC will be the direction of one of the sides of the triangle. In BC take $BD =$ the sum of the sides, join AD, and make $\angle DAE = \angle ADB$, then AEB will be the triangle required; for $\angle ADE = \angle DAE$, $\therefore AE = DE$, and $AE + EB = DB$ the sum of the sides.

5. $DECF$ (fig. 22) is evidently a parallelogram, and $\angle CDE = \angle DCF = \angle DCE$; $\therefore DE = CE = DF$.

6. Let A (fig. 23) be the common centre of the two circles, B the given point; BPQ a line drawn through B cutting the circles in P and Q; join AP, AQ and let $BP = \rho$, $BQ = \rho'$,

$$\angle ABQ = \theta, \ AP = a, \ AQ = 2a, \ AB = c;$$

$$\therefore \rho^2 + c^2 - 2c\rho\cos\theta = a^2; \ \rho'^2 + c^2 - 2c\rho'\cos\theta = 4a^2;$$

and if $\rho^1 = 2\rho$, we have $4\rho^2 + c^2 - 4c\rho\cos\theta = 4a^2$,

but $4\rho^2 + 4c^2 - 8c\rho\cos\theta = 4a^2$;

$$\therefore\ 4c\rho\cos\theta = 3c^2 \text{ or } \rho\cos\theta = \frac{3c}{4}.$$

Hence in AB take $BM = \frac{3}{4}AB$ or $AM = \dfrac{AB}{4}$, draw MP perpendicular to AB, meeting the inner circle in P; join BPQ, and BP will be equal to PQ.

7. Let O (fig. 24) be the centre, $\angle CAO = \theta$, $AO = a$; then $CA = 2a\cos\theta$, and $CP = n\,(CA)$;

$$\therefore\ AP = (n+1)\,CA = 2\,(n+1)\,a\cos\theta,$$

or the polar equation to the locus of P is $\rho = 2\,(n+1)\,a\cos\theta$, which is the equation to a circle whose centre is in the line AO, and radius $= (n+1)\,a$.

8. Draw DK (fig. 25) touching the circle at D; then

$$\angle\,DFG = \angle\,KDE = \angle\,EHG:$$

hence a circle may be described through the four points F, E, G, H since EFG and EHG will be angles in the same segment and upon the same base EG, and are proved to be equal to one another.

9. (Fig. 26) $\dfrac{Ac}{AO} = \dfrac{\sin AOc}{\sin AcO}$; $\dfrac{Bc}{BO} = \dfrac{\sin BOc}{\sin BcO}$;

$$\therefore\ \frac{Ac}{Bc} = \frac{AO}{BO}\cdot\frac{\sin AOc}{\sin BOc}.$$

Similarly $\dfrac{Ba}{Ca} = \dfrac{BO}{CO}\cdot\dfrac{\sin BOa}{\sin COa}$,

$$\frac{Cb}{Ab} = \frac{CO}{AO}\cdot\frac{\sin COb}{\sin AOb};$$

hence by multiplication

$$\frac{Ac\,.\,Ba\,.\,Cb}{Bc\,.\,Ca\,.\,Ab} = 1, \text{ or } Ac\,.\,Ba\,.\,Cb = Bc\,.\,Ca\,.\,Ab.$$

10. Let PQ (fig. 27) be the straight line whose equation is required, C the centre, r the radius of the given circle; CA

the axis of x; $\angle PCA = \alpha$, $\therefore \angle QCA = 90 + \alpha$; and the co-ordinates of the points P, Q are $a \cos \alpha$, $a \sin \alpha$; and $-a \sin \alpha$, $a \cos \alpha$ respectively; therefore the equation to PQ is

$$y - a \sin \alpha = \frac{a \cos \alpha - a \sin \alpha}{-a \sin \alpha - a \cos \alpha} (x - a \cos \alpha)$$

$$= \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha} (x - a \cos \alpha);$$

$$\therefore\ y = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha} x + \frac{a}{\sin \alpha + \cos \alpha}.$$

hence $$y = \tan (\alpha - 45)\, x + \frac{a}{\sqrt{2}} \sec (\alpha - 45)$$

is the equation required.

11. Let h, k be the co-ordinates of the given point without the circle; x', y', x'', y'' the co-ordinates of the two points of contact, then the equations to the two tangents are

$$x'x + y'y = a^2, \text{ and } x''x + y''y = a^2;$$

and since they both pass through a point whose co-ordinates are h, k, we have $hx' + ky' = a^2$, $hx'' + ky'' = a^2$;

and the equation to the line joining the points of contact is

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x'); \text{ but } \frac{y'' - y'}{x'' - x'} = -\frac{h}{k};$$

$$\therefore\ y - y' = -\frac{h}{k} (x - x') \text{ or } hx + ky = hx' + ky' = a^2$$

the equation required.

12. Let ACB (fig. 28) be the right-angled triangle, D the middle point of the hypothenuse AB;

$$AC = a_1,\ BC = a_2,\ AB = c;$$

$$\therefore\ \delta_1 = 2 \frac{a_1}{2} \tan \frac{A}{2} = a_1 \tan \frac{A}{2}; \quad \delta_2 = a_2 \tan \frac{B}{2},$$

$$\text{or } \frac{1}{\delta_1} - \frac{1}{\delta_2} = \frac{\cot \frac{A}{2}}{a_1} - \frac{\cot \frac{B}{2}}{a_2} = \frac{1}{c} \left(\frac{\cot \frac{A}{2}}{\sin B} - \frac{\cot \frac{B}{2}}{\sin A} \right)$$

$$= \frac{1}{c}\left(\frac{2\cos^2\frac{A}{2} - 2\cos^2\frac{B}{2}}{\sin A \sin B}\right) = \frac{1}{c}\frac{(\cos A - \cos B)}{\sin A \sin B}$$

$$= \frac{c\,(\cos A - \cos B)}{(c \sin A)\, c \sin B} = \frac{a_1 - a_2}{a_2 a_1};$$

$$\therefore\ \frac{1}{\delta_1} - \frac{1}{\delta_2} = \frac{a_1 - a_2}{a_1 a_2} = \frac{1}{a_2} - \frac{1}{a_1}.$$

13. Let α, β be the co-ordinates of the given point P (fig. 29), referred to the two sides AB, AC as axes; DPE the required straight line meeting AB, AC in the points D, E; then if $AD = m$, $AE = n$, the equation to DPE is $\frac{x}{m} + \frac{y}{n} = 1$, and since this passes through the point P whose co-ordinates are α, β, we have $\frac{\alpha}{m} + \frac{\beta}{n} = 1$,

$$\text{and } \triangle EAD = \frac{mn}{2}\sin A = \tfrac{1}{2}\triangle CAB = \tfrac{1}{2}\left(\frac{a^2}{2}\sin A\right);$$

$$\therefore\ mn = \frac{a^2}{2}, \text{ hence } \alpha n + \beta m = mn = \frac{a^2}{2},$$

$$\text{and } \alpha n - \beta m = \pm\sqrt{(\alpha n + \beta m)^2 - 4\alpha\beta mn}$$

$$= \pm\sqrt{\frac{a^4}{4} - 2\alpha\beta a^2} = \pm\frac{a^2}{2}\sqrt{1 - \frac{8\alpha\beta}{a^2}};$$

$$\therefore\ 2\alpha n = \frac{a^2}{2}\left(1 \pm \sqrt{1 - \frac{8\alpha\beta}{a^2}}\right),$$

$$2\beta m = \frac{a^2}{2}\left(1 \mp \sqrt{1 - \frac{8\alpha\beta}{a^2}}\right), \text{ and } mn = \frac{a^2}{2};$$

$$\therefore\ \frac{1}{m} = \frac{1}{2\alpha}\left(1 \pm \sqrt{1 - \frac{8\alpha\beta}{a^2}}\right),\quad \frac{1}{n} = \frac{1}{2\beta}\left(1 \mp \sqrt{1 - \frac{8\alpha\beta}{a^2}}\right),$$

and the equation to DPE becomes

$$\frac{1}{2\alpha}\left(1 \pm \sqrt{1 - \frac{8\alpha\beta}{a^2}}\right)x + \frac{1}{2\beta}\left(1 \mp \sqrt{1 - \frac{8\alpha\beta}{a^2}}\right)y = 1.$$

ST. JOHN'S COLLEGE. Dec. 1833. (No. IV.)

1. Find the centre of a given circle, and state where Euclid's demonstration is imperfect. What is the meaning of a given circle in Analytical Geometry?

2. If a straight line be at right angles to a plane, every plane in which the straight line lies shall be at right angles to that plane.

3. Inscribe the greatest quadrilateral figure in a given circle. Can a circle always be inscribed in a proposed quadrilateral figure, or described about one?

4. Given a polygon traced upon a plane, describe the triangle that shall have an equivalent area.

5. ACB is an isosceles triangle having a right angle C; with centre C and distance CA describe a circle; if from a point Q in the circumference of the circle, QRr be drawn parallel to AB meeting AC, BC in R, r respectively, prove that $QR^2 + Qr^2 = AB^2$.

6. Inscribe in a circle a triangle whose sides or sides produced shall pass through three given points in the same plane.

7. ABC is a triangle inscribed in a circle; AB, AC, BC produced cut a line given in position in the points m, n, p respectively. If t_m, t_n, t_p be the lengths of the lines drawn from m, n, p touching the circle, shew that

$$t_m t_n t_p = An \,.\, Bm \,.\, Cp = Am \,.\, Bp \,.\, Cn.$$

8. An indefinite number of straight lines (not in the same plane) are situated so that from a fixed point perpendiculars can be drawn to them which are all equal to one another: required the locus of the intersections of these perpendiculars with the given lines.

9. Given a circle traced upon a plane, describe another whose area is exactly twice as great as the former.

10. Investigate the line or lines represented by the equation

$$y^3 + (x - a)\, y^2 + (x^2 - a^2)\, y + (x - a)^2 (x + a) = 0.$$

11. Taking the requisite data to fix a parallelogram in a plane by equations to its sides; prove that the diagonals bisect each other.

12. Given the equation to a circle and the chord of the circle; shew that a perpendicular let fall from the centre of the circle upon the chord, bisects the chord.

13. One of the vertices of a triangle being taken for the origin of rectangular co-ordinates, and x', y', x'', y'' the co-ordinates of the other two; prove that the area of the triangle $= \frac{1}{2}(x'y'' - y'x'')$.

SOLUTIONS TO (No. IV.)

1. Euclid, Prop. 1, Book III. See Potts' Euclid, Page 107.

In Analytical Geometry if the equation to a circle is given, the position of the centre and radius may be determined, and the circle is therefore determined in magnitude and position.

2. Euclid, Prop. 18, Book XI.

3. (a) Let $ABCD$ (fig. 30) be a quadrilateral figure; draw the diagonals AC, BD intersecting in E; then if α be the angle between the diagonals, $\triangle ABC = \frac{1}{2} AC . BE . \sin\alpha$: Also $\triangle ADC = \frac{1}{2} AC . DE . \sin\alpha$; therefore by addition, the area of the quadrilateral $ABCD = \frac{1}{2}(AC . BD . \sin\alpha)$: this will be the greatest when AC, BD and $\sin\alpha$ are respectively greatest. Now if the quadrilateral be inscribed in a circle, AC, BD are the greatest possible when they are two diameters; and $\sin\alpha$ is greatest, when AC, BD are at right angles. Hence if two diameters AC, BD be drawn at right angles, and AB, BC, CD, DA be joined, the square $ABCD$ will be the greatest possible quadrilateral figure which can be inscribed in the circle.

(β) When a quadrilateral figure is inscribed in a circle, the opposite angles are together equal to two right angles; if this condition be not satisfied the quadrilateral figure cannot be inscribed in a circle.

(γ) If a circle can be inscribed in $ABCD$ (fig. 31) touching the sides AB, BC, CD, DA in the points a, b, c, d respectively, then

$$Aa = Ad,\ Ba = Bb,\ Dc = Dd,\ Cc = Cb,$$

$$\therefore\ Aa + Ba + Dc + Cc = Ad + Dd + Bb + Cb,$$

$$\text{or}\ AB + CD = BC + DA;$$

hence the sum of two opposite sides is equal to the sum of the two remaining sides.

4. Let the parallelogram $ABCD$ (fig. 32) be described

equal to the given rectilineal figure; produce AB to E making $BE = AB$; join AC, CE; then

$$\Delta ACE = 2\Delta ACB = \square\, ABCD$$

equal the given rectilineal figure.

5. Draw CDN (fig. 33) perpendicular to the base, meeting AB, Rr in D and N respectively; then since Rr is bisected in N, $RQ^2 + Qr^2 = 2(RN^2 + QN^2) = 2(CN^2 + NQ^2)$

$$(\text{since } \angle CRN = \tfrac{1}{2} \text{ a right angle} = \angle RCN)$$

$$= 2CA^2 = CA^2 + CB^2 = AB^2.$$

6. See Appendix I. Cor. Art. 1.

7. Let $\angle Amp$ (fig. 34) $= \theta$, $\angle Anp = \phi$, $\angle Bpm = \psi$,

$$\text{then} \quad \frac{Am}{An} = \frac{\sin\phi}{\sin\theta}, \quad \frac{Bp}{Bm} = \frac{\sin\theta}{\sin\psi}, \quad \frac{Cn}{Cp} = \frac{\sin\psi}{\sin\phi};$$

therefore by multiplication, $\dfrac{Am \,.\, Bp \,.\, Cn}{An \,.\, Bm \,.\, Cp} = 1$,

$$\text{also } t_m^2 = Am \,.\, Bm, \quad t_n^2 = Cn \,.\, An, \quad t_p^2 = Bp \,.\, Cp;$$

$$\therefore\ t_m^2 \,.\, t_n^2 \,.\, t_p^2 = (Am \,.\, Bp \,.\, Cn)(An \,.\, Bm \,.\, Cp) = (Am \,.\, Bp \,.\, Cn)^2,$$

$$\text{or } t_m \,.\, t_n \,.\, t_p = Am \,.\, Bp \,.\, Cn = An \,.\, Bm \,.\, Cp.$$

8. Let the fixed point A be taken for the origin, P any point in the locus; then since AP is constant, it is evident that the locus of P is a sphere round the centre A, and radius $= AP$.

9. Find C (fig. 35) the centre of the circle; draw any radius CA, and AB perpendicular to CA and equal to it; join CB and with centre C and radius CB describe a circle, this will be the circle required.

For $CB^2 = CA^2 + AB^2 = 2CA^2$, and circles are to one another as the squares of their radii; therefore the circle whose radius is CB : circle whose radius is CA :: $CB^2 : CA^2$ :: $2 : 1$.

10. The equation may be reduced to the form

$$(y + x - a)(y^2 + x^2 - a^2) = 0,$$

which is satisfied by making separately

$$y + x - a = 0, \text{ and } y^2 + x^2 - a^2 = 0.$$

If CA, CB (fig. 36) be taken for the co-ordinate axes, the former equation represents the straight line AB, where $CA = CB = a$, and the latter the circle ABD whose centre is C and radius CA. Hence the proposed equation represents the circle ABD, and the chord of the quadrant AB.

11. Let the two sides AB, AD (fig. 37) of the parallelogram $ABCD$ be taken for the co-ordinate axes; join AC, BD, intersecting each other in E, draw EF parallel to AD, and let $AB = a$, $AD = b$; then the equations to the diagonals AC, BD are $y = \dfrac{bx}{a}$, and $\dfrac{x}{a} + \dfrac{y}{b} = 1$, and if x', y' be the co-ordinates of their point of intersection E,

$$y' = \frac{bx'}{a}, \quad \frac{x'}{a} + \frac{y'}{b} = 1;$$

$$\therefore \frac{2x'}{a} = 1, \text{ or } x' = AF = \frac{a}{2}; \text{ and } \frac{2y'}{b} = 1, \text{ or } y' = EF = \frac{b}{2},$$

$$\text{hence } \frac{AE}{AC} = \frac{AF}{AB} = \tfrac{1}{2}, \text{ or } AE = \frac{AC}{2},$$

$$\text{and } \frac{BE}{BD} = \frac{EF}{AD} = \tfrac{1}{2}, \text{ or } BE = \frac{BD}{2};$$

and the diagonals of the parallelogram bisects each other.

12. Let the equation to the circle referred to the centre C (fig. 38) be $x^2 + y^2 = a^2$, and the equation to the chord PQ, $y = mx + c$; then the equation to CR drawn through C perpendicular to PQ is $y = -\dfrac{1}{m}x$, and if X, Y be the co-ordinates of the point R,

$$Y = -m^2 Y + c, \text{ or } Y = \frac{c}{1 + m^2} \text{ and } X = -\frac{cm}{1 + m^2};$$

and to find the co-ordinates of the points of intersection of PQ with the circle, we have

$$x^2 + (mx + c)^2 = a^2, \text{ or } (1 + m^2)\,x^2 + 2mc\,x + (c^2 - a^2) = 0\,;$$

therefore if x', x'' be the two values of x which are the abscissæ of the points P, Q

$$x' + x'' = -\frac{2mc}{1 + m^2} = 2X,$$

or R is the middle point of PQ.

13. Let ABC (fig. 39) be the triangle, Ax, Ay the co-ordinate axes; $\angle BAx = \theta'$, $CAx = \theta''$, $AB = \rho'$, $AC = \rho''$; then the area of the triangle ABC

$$= \tfrac{1}{2} AB\,.\,AC\,.\,\sin \angle BAC = \tfrac{1}{2}\rho'\rho''\sin(\theta'' - \theta')$$
$$= \tfrac{1}{2}(\rho''\sin\theta''\,.\,\rho'\cos\theta' - \rho''\cos\theta''\,.\,\rho'\sin\theta')$$
$$= \tfrac{1}{2}(y''x' - x''y').$$

ST JOHN'S COLLEGE. Dec. 1834. (No. V.)

1. What objections have been urged against the doctrine of parallel straight lines as it is laid down by Euclid? Where does the difficulty originate, and what has been suggested to remove it?

2. Magnitudes have the same ratio to one another which their equimultiples have. When is the first of four magnitudes said to have to the second the same ratio which the third has to the fourth; and when a greater ratio?

Do the definitions and theorems of Book v. include incommensurable magnitudes?

3. If a solid angle be contained by three plane angles, any two are together greater than the third.

Define the inclination of a plane to a plane, and shew that it is equal to the inclination of their normals.

4. If there be two concentric circles, and any chord of the greater circle cut the less in any point, this point will divide the chord into two segments whose rectangle is invariable.

5. Divide *algebraically* a given line (a) into two parts such that the rectangle contained by the whole and one part may be equal to the square of the other. Deduce Euclid's construction from one solution, and explain the other.

6. Find a straight line, which shall have to a given straight line the ratio of $1 : \sqrt{5}$.

7. ACB is a triangle whose base AB is divided in E and produced to F, so that $AE : EB$ and also $AF : FB$ as $AC : CB$. Join CE, CF and shew that $\angle ECF$ is a right angle.

8. The point C is the centre of a given circle, and E is any point in the radius; find that point in the circumference where CE subtends the greatest angle.

9. Two points are taken in the diameter of a circle at any equal distances from the centre; through one of these draw any chord, and join its extremities and the other point. The triangle so formed has the sum of its sides invariable.

10. If p_1, p_2, p_3 be perpendiculars from any point within a triangle on the sides; P_1, P_2, P_3 perpendiculars from the angular points on the same sides respectively, prove that

$$\frac{p_1}{P_1} + \frac{p_2}{P_2} + \frac{p_3}{P_3} = 1.$$

11. *MANP* is a parallelogram having a given angle at *A*, and also its perimeter a given quantity. Find the locus of *P* for all such parallelograms and construct it.

12. Find the locus of a point such that if straight lines be drawn from it to the four corners of a given square, the sum of their squares shall be invariable.

13. Given the equations to two straight lines passing through two given points, find the locus of their point of concourse when the straight lines intersect each other at a given angle.

14. In any parallelopiped, the sum of the squares of the four diagonals is equal to the sum of the squares of the twelve edges.

15. If a triangular pyramid have one of its solid angles a right angle, i. e. contained by three plane right angles, the square of the face subtending the right angle is equal to the squares of the three faces which contain it.

SOLUTIONS TO (No. V.)

1. See Potts' Euclid, p. 50.

2. Euclid, Prop. 15. Book v. and Definitions 5 and 7 Book v.; and Potts' Euclid, note to Definition 5. Book. v. page 162.

3. Euclid, Prop. 20. Book xi. and Definition 6. Book xi.

Let A (fig. 40) be a point in the common section AB of the two planes, from which draw AC, AD at right angles to AB in the two planes; draw also AE, EF perpendicular to the two planes BAC, BAD respectively; then AE is perpendicular to the plane BAC and therefore to AB. Hence AB is perpendicular to AE, AC, AD which are consequently in the same plane. Similarly AF may be proved to be in the same plane with AC and AD; and right $\angle EAC$ = right $\angle FAD$; $\therefore$ $\angle EAF = \angle DAC$.

4. Let C (fig. 41) be the common centre of the two circles; $PpqQ$ a chord cutting the inner circle in p and q; draw CD perpendicular to PQ, it will bisect PQ and pq in the point D; join CP, Cp; then

$$Qp.Pp = PD^2 - pD^2 = (PC^2 - CD^2) - (pC^2 - CD^2) = CP^2 - Cp^2,$$

and is therefore invariable.

5. See Potts' Euclid, p. 73.

Taking Euclid's figure,

$$AH = AF = \frac{\sqrt{5} - 1}{2} a;$$

$$\therefore\ EF = \frac{\sqrt{5}}{2}.a = \sqrt{a^2 + \frac{a^2}{4}} = \sqrt{AB^2 + AE^2} = BE,$$

which gives Euclid's construction.

6. Let AB (fig. 43) be the given straight line; draw BC at right angles to AB and $= 2AB$, join AC, and draw BD perpendicular to AC; then AD will be the line required.

$$\text{For} \quad AC = \sqrt{AB^2 + BC^2} = \sqrt{5}. \quad AB,$$

$$\text{and} \quad AD : AB :: AB : AC :: 1 : \sqrt{5}.$$

7. Produce AC (fig. 43) to G; then CE, CF bisect the angles ACB, BCG respectively;

$$\therefore \angle ECF = \angle BCF + \angle BCE = \tfrac{1}{2}(\angle BCG + \angle BCA)$$

equal a right angle.

8. Draw EP (fig. 44) perpendicular to EC meeting the circle in P: on CP as a diameter describe a semicircle which will touch the circle in P and pass through E, because $\angle PEC$ is a right angle. Draw any line $CQ'Q$ meeting the circles in Q', Q; join EQ, EQ'; then $\angle CPE = \angle CQ'E$ is greater than $\angle CQE$, or $\angle CPE$ is greater than any other angle subtended by CE at a point in the circumference of the given circle.

9. Let C (fig. 45) be the centre of the circle, D, E two points in the diameter AB at equal distances from C; PDQ any chord through D; join CP, CQ, EP, EQ; then

$$PD^2 + PE^2 = 2\,CP^2 + 2\,CD^2 = 2\,AC^2 + 2\,CD^2;$$

$$DQ^2 + QE^2 = 2\,CQ^2 + 2\,CD^2 = 2\,AC^2 + 2\,CD^2;$$

$$2\,DQ\,.\,DP = 2\,AD\,.\,DB \qquad = 2\,AC^2 - 2\,CD^2;$$

therefore by addition

$$PQ^2 + PE^2 + QE^2 = 6\,AC^2 + 2\,CD^2,$$

and is invariable for every position of the chord PQ passing through D.

10. Let P (fig. 46) be the point within the triangle ABC; join AP, BP, CP; then

$$\triangle BPC = p_1 \frac{BC}{2} = \frac{p_1}{P_1} \cdot P_1 \cdot \frac{BC}{2} = \frac{p_1}{P_1} \triangle ABC;$$

$$\text{similarly} \quad \triangle APC = \frac{p_2}{P_2} \triangle ABC;$$

$$\text{and} \quad \triangle APB = \frac{p_3}{P_3} \triangle ABC;$$

therefore by addition

$$\triangle BPC + \triangle APC + \triangle APB = \triangle ABC = \left(\frac{p_1}{P_1} + \frac{p_2}{P_2} + \frac{p_3}{P_3}\right) \triangle ABC;$$

$$\text{or} \quad \frac{p_1}{P_1} + \frac{p_2}{P_2} + \frac{p_3}{P_3} = 1.$$

11. Let AM, AN (fig. 47) be taken for the axes of x and y; and x, y the co-ordinates of P; then $x + y = \frac{1}{2}$ the perimeter of the parallelogram $= \frac{p}{2}$; or the locus of P is a straight line cutting the axes of x and y at distances AB, $AC = \frac{p}{2}$ from A.

12. Let $ABCD$ (fig. 48) be the given square whose side $AB = a$; P a point in the required locus, draw PM perpendicular to AB, and let $AM = x$, $PM = y$; then

$$AP^2 = x^2 + y^2, \quad BP^2 = (a - x)^2 + y^2,$$

$$DP^2 = x^2 + (a - y)^2, \quad CP^2 = (a - x)^2 + (a - y)^2;$$

$$\therefore AP^2 + BP^2 + DP^2 + CP^2 = 2\{x^2 + (a - x)^2 + y^2 + (a - y)^2\} = c^2;$$

$$\therefore 2x^2 - 2ax + 2y^2 - 2ay = \frac{c^2 - 4a^2}{2}$$

$$\text{and} \quad \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{c^2 - 2a^2}{4},$$

the equation to a circle the co-ordinates of whose centre are

$$\frac{a}{2}, \frac{a}{2}, \text{ and radius} = \frac{1}{2}\sqrt{c^2 - 2a^2}.$$

Hence in order that the problem may be possible c^2 must not be less than $2a^2$.

13. Let the straight line joining the two points A, B (fig. 49) be taken for the axis of x; C, the middle point of AB, the origin; $AC = c$; then the equations to the two lines AP, BP are $y = m(x + c)$, $y = m'(x - c)$ respectively. And if (α) be the angle between them;

$$\tan\alpha = \frac{m' - m}{1 + mm'} = \frac{\dfrac{y}{x - c} - \dfrac{y}{x + c}}{1 + \dfrac{y^2}{x^2 - c^2}} = \frac{2cy}{x^2 + y^2 - c^2};$$

$\therefore\ x^2 + y^2 - c^2 = 2c\cot\alpha\, y$, and $x^2 + (y - c\cot\alpha)^2 = (c\operatorname{cosec}\alpha)^2$; which is the equation to a circle whose centre is in the axis of y at a distance $c\cot\alpha$ from the origin, and radius $= c\operatorname{cosec}\alpha$.

14. Let AB, BC, CD, DA (fig. 50) be the four sides of the base of the parallelopiped; ab, bc, cd, da the corresponding sides of the top; then $ACca$ will form a parallelogram whose sides are two diagonals AC, ca of the base and top, and the two sides Aa, Cc of the parallelopiped; and the diagonals Ac, Ca of this parallelogram will be two diagonals of the parallelopiped. Similarly, $BDdb$ will form a parallelogram having its two diagonals Bd, Db the two remaining diagonals of the parallelopiped. Now if $AB = a$, $AD = b$, $\angle BAD = \alpha$; $BD = d$, $AC = d'$, we have

$$d^2 = a^2 + b^2 - 2ab\cos\alpha, \text{ and } d'^2 = a^2 + b^2 + 2ab\cos\alpha;$$

$$\therefore\ d^2 + d'^2 = 2(a^2 + b^2),$$

or in any parallelogram, the sum of the squares of the diagonals is double the sum of the squares of the sides. Hence

$$Ac^2 + Ca^2 = 2(AC^2 + Cc^2),$$

$$Bd^2 + Db^2 = 2(BD^2 + Bb^2) = 2(BD^2 + Cc^2);$$

$$\therefore\ Ac^2 + Ca^2 + Bd^2 + Db^2 = 2(AC^2 + BD^2) + 4Cc^2$$

$$= 2\{2(AB^2 + BC^2)\} + 4Cc^2 = 4(AB^2 + BC^2 + Cc^2)$$

= the sum of the squares of the twelve edges.

15. Let O (fig. 51) be the vertex of the pyramid; OA, OB, OC the three sides at right angles to one another $= a, b, c$ respectively: then

$$AB = \sqrt{a^2+b^2}, \quad AC = \sqrt{a^2+c^2}, \quad BC = \sqrt{b^2+c^2},$$

and $$\cos \angle BAC = \frac{AC^2 + AB^2 - BC^2}{2AB \,.\, AC} = \frac{a^2}{\sqrt{(a^2+b^2)(a^2+c^2)}};$$

$$\therefore \ \sin \angle BAC = \frac{\sqrt{a^2b^2 + a^2c^2 + b^2c^2}}{\sqrt{(a^2+b^2)(a^2+c^2)}},$$

and $$\triangle ABC = \tfrac{1}{2} AB \,.\, AC \,.\, \sin \angle BAC = \tfrac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2};$$

$$\text{or } (\triangle ABC)^2 = \left(\frac{ab}{2}\right)^2 + \left(\frac{ac}{2}\right)^2 + \left(\frac{bc}{2}\right)^2$$

$$= (\triangle AOB)^2 + (\triangle AOC)^2 + (\triangle BOC)^2.$$

ST JOHN'S COLLEGE. Dec. 1835. (No. VI.)

1. In some treatises on Geometry it is laid down as an axiom more evident than Euclid's 12th, that two straight lines which cut one another, cannot both be parallel to the same straight line. Shew that this is only a disguise of Euclid's axiom.

Give an instance to shew how some of the fundamental theorems of Geometry may be proved *a priori* from considerations purely analytical. Two solid angles may be unequal, which are contained by the same number of equal angles in the same order.

2. If four magnitudes be proportionals, they shall also be proportionals when taken alternately. Prove by taking equimultiples according to Euclid's definition, that the magnitudes 4, 5, 7, 9 are not proportional.

3. Similar triangles are to one another in the duplicate ratio of their homologous sides. How does it appear from Euclid that the duplicate ratio of two magnitudes is the same as that of their squares?

4. A chord POQ cuts the diameter of a circle in O in an angle equal half a right angle; shew that

$$PO^2 + OQ^2 = 2\,(\text{rad.})^2.$$

5. Two circles touch internally; describe another of given radius (not greater than the difference of the radii of the former) so as to touch both. Prove that the locus of the centres of all the circles so described, is an ellipse whose foci are the centres of the two given circles.

6. If the lines bisecting the vertical angles of a triangle be divided into parts which are to one another as the base to the sum of the sides, the point of division is the centre of the inscribed circle.

7. Find the locus of the points of *quadrisection* of all parallel chords in a circle; employing the equations

$$y^2 = a^2 - x^2; \; y = ax + \beta.$$

8. If all the ordinates of a circle be moved through a given angle, the abscissa and magnitude of each ordinate remaining the same; what will be the curve, the origin of co-ordinates being the centre?

9. Eliminate (a) between the equations

$$y^2 = a(x - a); \; y^2 = (a + \delta)(x - a - \delta).$$

Explain the geometrical meaning of the result, and trace the change as (δ) diminishes and ultimately vanishes.

10. In the hyperbola, at a point P whose ordinate

$$= \frac{(BC)^{\frac{3}{2}}}{(CS)^{\frac{1}{2}}},$$

prove that $Py = CS$, Cy being a perpendicular from the centre on the tangent at P.

11. If about the exterior focus of a hyperbola, a circle be described with radius equal the semi-conjugate axis, and tangents be drawn to it from any point in the hyperbola, the line joining the points of contact will touch the circle described upon the transverse axis as its diameter.

12. If PSQ be a focal chord of an ellipse, tangents at P and Q will meet in the directrix.

13. In an ellipse the sum of the squares of two conjugate normals is constant.

14. S is any point in the diameter AB of a circle whose circumference is divided into $2n$ equal parts; if lines be drawn from S to all points of section, the sum of their squares

$$= n(SA^2 + SB^2).$$

15. A sphere is described touching a face of any triangular pyramid and the other three produced, and three other spheres in like manner touching the remaining faces. If r_1, r_2, r_3, r_4 be their radii, and r the radius of the inscribed sphere, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{r}$.

SOLUTIONS TO (No. VI.)

1. SEE Potts' Euclid, p. 50. Also Note to Prop. 6, Book I. p. 50; and Note to Def. 9, Book XI. p. 253.

2. Euclid, Prop. 11, Book V.

Let the equimultiples of the first and third be 5, and of the second and fourth be 4; then the multiples of the first, second, third and fourth are respectively 20, 20, 35, 36; or the multiple of the first is equal to the multiple of the second, but the multiple of the third is not equal to the multiple of the fourth; and the four quantities are therefore not proportionals.

Similarly, if the equimultiples of the first and third be 19, and the equimultiples of the second and fourth be 15, the multiples of the first, second, third and fourth respectively become, 76, 75, 133, 135; or the multiple of the first is greater than the multiple of the second, but the multiple of the third not greater than the multiple of the fourth; hence the first has a greater ratio to the second than the third has to the fourth.

3. Euclid, Prop. 19, Book I.

If $A : B :: B : C$, then the ratio of A to C is that compounded of the ratios of A to B and of B to C, or of A to B and A to B, and is therefore the same of the ratio of A^2 to B^2.

4. From the centre C, (fig. 52) draw CN perpendicular to PQ; then since PQ is bisected in N,

$$PO^2 + QO^2 = 2(QN^2 + NO^2) = 2(QN^2 + NC^2),$$

(since $\angle OCN = \angle PNC - \angle POC = \frac{1}{2}$ a right angle $= \angle NOC$);

$$\therefore\ PO^2 + QO^2 = 2QC^2.$$

5. Let C, O (fig. 53) be the centres of the two circles whose radii are a, b; P the centre of the circle which touches both circles; ρ its radius; then $OP = b + \rho$; $CP = a - \rho$; hence if on the base OC a triangle OPC be described whose two sides OP, CP are equal to $b + \rho$, $a - \rho$ respectively, the vertex P will be the centre of the required circle. Also

$$OP + CP = b + \rho + a - \rho = a + b;$$

hence the locus of P is an ellipse whose foci are O, C; and axis major $= a + b$.

6. Let CD (fig. 54) bisect the angle ACB; divide CD in E, so that $DE : EC :: AB : AC + CB$, and draw CM, EN perpendicular to AB; then

$$\frac{DE}{EC} = \frac{c}{a+b}, \therefore \frac{DE}{DC} = \frac{EN}{CM} = \frac{c}{a+b+c};$$

hence $c\,(CM) = (a + b + c)\,EN$. But if r be the radius of the inscribed circle,

$$(a + b + c)\,r = 2 \text{ area of } \triangle ABC = c\,.\,CM = (a + b + c)\,EN;$$

$\therefore$ $EN = r$; and the centre of the inscribed circle lies in the line CD, therefore E is the centre of the circle inscribed in the triangle ABC.

7. Transform the origin to a point X, Y in one of the parallel chords; then $(Y + y)^2 + (X + x)^2 = a^2$, and $y = \alpha x$ are the equations to the circle and chord respectively. Hence at the points of intersection of the circle and chord,

$$(Y + \alpha x)^2 + (X + x)^2 = a^2. \quad (1).$$

Now since the origin is in the quadrisection of the chord, if x' be one value of x derived from equation (1), $-3x'$ will be the other; therefore the equation must be of the form

$$(x - x')\,(x + 3x') = 0, \text{ or } x^2 + 2x'x - 3x'^2 = 0;$$

$$\text{but } (1 + \alpha^2)\,x^2 + 2\,(\alpha Y + X)\,x + X^2 + Y^2 - a^2 = 0;$$

$$\therefore\ x' = \frac{\alpha Y + X}{1 + \alpha^2};\quad 3\,x'^2 = \frac{a^2 - (X^2 + Y^2)}{1 + \alpha^2};$$

$$\text{or } 3\left(\frac{\alpha Y + X}{1 + \alpha^2}\right)^2 = \frac{a^2 - (X^2 + Y^2)}{1 + \alpha^2};$$

hence $(4 + \alpha^2)\,X^2 + 6\alpha XY + (4\alpha^2 + 1)\,Y^2 = (1 + \alpha^2)\,a^2$ is the equation to the locus required, and is that of an ellipse whose centre coincides with the centre of the circle.

8. Let x', y' be the co-ordinates of any point in the required curve, $90 - \alpha$ the angle through which the ordinates are

moved, x, y, the co-ordinates of the point in the circle corresponding to the point x', y'; then

$$x = x' - y' \cot \alpha\,; \quad y = y' \operatorname{cosec} \alpha\,;$$

$$\text{and } x^2 + y^2 = a^2\,; \quad \therefore \; (x' - y' \cot \alpha)^2 + (y' \operatorname{cosec} \alpha)^2 = a^2,$$

$$\text{or } x'^2 - 2x'y' \cot \alpha + y'^2 \left\{1 + 2(\cot \alpha)^2\right\} = a^2,$$

the equation to an ellipse, whose centre coincides with the centre of the circle. Let the curve be transformed to polar co-ordinates by putting $x' = \rho \cos \theta$, $y' = \rho \sin \theta$;

$$\therefore \; \rho^2 \left\{\cos^2 \theta - \sin 2\theta \,.\, \cot \alpha + (1 + 2 \cot^2 \alpha) \sin^2 \theta\right\} = a^2,$$

$$\text{or } \rho^2 \left\{1 + \cot^2 \alpha \,(1 - \cos 2\theta) - \sin 2\theta \cot \alpha\right\} = a^2\,;$$

$$\therefore \; \rho^2 \left\{\operatorname{cosec}^2 \alpha - \cot \alpha \operatorname{cosec} \alpha \cos (2\theta - \alpha)\right\} = a^2\,;$$

$$\text{and } \rho^2 = \frac{a^2 \sin^2 \alpha}{1 - \cos \alpha \cos (2\theta - \alpha)} = \frac{2a^2 \sin^2 \dfrac{\alpha}{2}}{1 - \cos \alpha \sec^2 \dfrac{\alpha}{2} \cos^2 \left(\theta - \dfrac{\alpha}{2}\right)},$$

which is the equation to an ellipse whose axes are

$$2a\sqrt{1 - \cos \alpha}, \text{ and } 2a\sqrt{1 + \cos \alpha},$$

and the inclination of the axis-major to the axis of $x = \dfrac{\alpha}{2}$.

9. The equations to the two curves are $y^2 = a(x - a)$, and $y^2 = a(x - a) + \delta(x - 2a) - \delta^2$; hence at their points of intersection $\delta(x - 2a) - \delta^2 = 0$,

$$\text{or } a = \frac{x - \delta}{2}, \text{ and } x - a = \frac{x + \delta}{2};$$

$$\therefore \; y^2 = a(x - a) = \frac{x^2 - \delta^2}{4}.$$

Hence for a given value of (δ), every pair of parabolas whose equations are $y^2 = a(x - a)$, $y^2 = (a + \delta)(x - a - \delta)$ intersect each other in the hyperbola whose equation is $y^2 = \dfrac{x^2 - \delta^2}{4}$; and whose axes are δ, 2δ.

As δ diminishes, the hyperbola $y^2 = \dfrac{x^2 - \delta^2}{4}$ approaches to

the two straight lines represented by the equations $y^2 = \frac{x^2}{4}$, or $y = \pm\frac{x}{2}$; hence the parabolas ultimately intersect in the two straight lines $y = \frac{x}{2}$, and $y = -\frac{x}{2}$. The straight lines manifestly touch all the parabolas represented by the equation $y^2 = a(x - a)$ when every possible value is assigned to a.

10. $$y^2 = \frac{b^3}{\sqrt{a^2 + b^2}}; \quad \therefore \frac{x^2}{a^2} = 1 + \frac{b}{\sqrt{a^2 + b^2}},$$

$$\text{or } x^2 = a^2 + \frac{a^2 b}{\sqrt{a^2 + b^2}};$$

hence $$CP^2 = x^2 + y^2 = a^2 + b\sqrt{a^2 + b^2};$$

$$CD^2 = CP^2 - a^2 + b^2 = b^2 + b\sqrt{a^2 + b^2};$$

$$\therefore CY^2 = PF^2 = \frac{a^2 b^2}{CD^2} = b\sqrt{a^2 + b^2} - b^2;$$

hence $PY^2 = CP^2 - CY^2 = a^2 + b^2$ or $PY = CS$.

11. Taking the centre of the hyperbola for the origin of co-ordinates, the equation to the circle whose centre is H and radius b is $(x + ae)^2 + y^2 = b^2$; and the equation to a tangent at the point x, y is

$$y' - y = -\frac{x + ae}{y}(x' - x),$$

or $$yy' + (x + ae)x' = y^2 + x(x + ae)$$

$$= b^2 - aex - a^2e^2 = -a^2 - aex;$$

now if this passes through a point h, k,

$$ky + (x + ae)h = -a^2 - aex,$$

$$\text{or } ky + (h + ae)x = -a^2 - aeh; \quad (1)$$

which is the equation to the straight line joining the two points

of contact, when tangents are drawn to the circle from the point h, k.

The equation to the circle described on the transverse axis is $x^2+y^2=a^2$; and the equation to a tangent at the point x_1, y_1 is $xx_1+yy_1=a^2$; and in order that this may coincide with equation (1), we must have

$$\frac{x_1}{a^2} = -\frac{h+ae}{a^2+aeh} \quad \text{or} \quad \frac{x_1}{a} = -\frac{h+ae}{a+eh};$$

$$\text{and} \quad \frac{y_1}{a^2} = -\frac{k}{a^2+aeh} \quad \text{or} \quad \frac{y_1}{a} = -\frac{k}{a+eh};$$

$$\therefore \frac{k^2+(h+ae)^2}{(a+eh)^2} = \left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{a}\right)^2 = 1;$$

$$\text{or} \quad k^2-(e^2-1)\,h^2+a^2\,(e^2-1)=0;$$

hence $\dfrac{h^2}{a^2}-\dfrac{k^2}{b^2}=1$, which shews that the point h, k is a point in the hyperbola.

12. Let tangents be drawn from a point whose co-ordinates measured from the centre are h, k; then the equation to the straight line joining the points of contact is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1;$$

let this pass through the focus; therefore when $x=ae$, $y=0$; hence $\dfrac{he}{a}=1$, or $h=\dfrac{a}{e}$: which shews that the point h, k lies in the directrix.

13. Let PK, DK' be two normals at the points P, D whose co-ordinates are x, y and x', y' respectively; then

$$PK^2 = y^2+\frac{b^4}{a^4}x^2; \quad DK'^2 = y'^2+\frac{b^4}{a^4}x'^2;$$

$$\text{and} \quad x' = -\frac{ay}{b}; \quad y' = \frac{bx}{a};$$

$$\therefore\ PK^2 + DK'^2 = \left(1 + \frac{b^2}{a^2}\right) y^2 + \frac{b^2}{a^2}\left(1 + \frac{b^2}{a^2}\right) x^2$$

$$= b^2 \left(1 + \frac{b^2}{a^2}\right) \left(\frac{y^2}{b^2} + \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2 + b^2),$$

and is constant.

14. Let C (fig. 55) be the centre, P_1, $P_2, \dots P_{2n}$ the points of division of the circumference;

$$\angle ACP_1 = \theta; \quad \therefore\ \angle ACP_2 = \theta + \frac{\pi}{n}; \quad \angle ACP_3 = \theta + \frac{2\pi}{n}, \text{ \&c.}$$

and

$$SP_1^2 = SC^2 + CP_1^2 - 2SC.CP_1 \cos\theta = SC^2 + CA^2 - 2SC.CA\cos\theta;$$

$$SP_2^2 \qquad = SC^2 + CA^2 - 2SC.CA\cos\left(\theta + \frac{\pi}{n}\right);$$

$$SP_3^2 \qquad = SC^2 + CA^2 - 2SC.CA\cos\left(\theta + \frac{2\pi}{n}\right);$$

$$SP_{2n}^2 \qquad = SC^2 + CA^2 - 2SC.CA\cos\left\{\theta + \frac{(2n-1)\pi}{n}\right\};$$

$$\therefore\ (SP_1^2 + SP_2^2 + \dots + SP_{2n}^2) = 2n(SC^2 + CA^2)$$

$$- 2SC.CA\left\{\cos\theta + \cos\left(\theta + \frac{\pi}{n}\right) + \dots + \cos\left(\theta + \frac{(2n-1)\pi}{n}\right)\right\}$$

$$= 2n(SC^2 + CA^2) - SC.CA \frac{\left\{\sin\left(2\pi + \theta - \frac{\pi}{n} + \frac{\pi}{2n}\right) - \sin\left(\theta - \frac{\pi}{2n}\right)\right\}}{\sin\frac{\pi}{2n}}$$

$$= 2n(SC^2 + CA^2) = n(AS^2 + SB^2)$$

since AB is bisected in C.

15. Let V be the volume of the pyramid; S_1, S_2, S_3, S_4 the areas of the four triangular faces opposite to the angular points A, B, C, D respectively; then

$$V = \frac{r}{3}(S_1 + S_2 + S_3 + S_4);$$

$$V = \frac{r_1}{3}(S_2 + S_3 + S_4 - S_1);$$

$$V = \frac{r_2}{3}(S_1 + S_3 + S_4 - S_2);$$

$$V = \frac{r_3}{3}(S_1 + S_2 + S_4 - S_3);$$

$$V = \frac{r_4}{3}(S_1 + S_2 + S_3 - S_4);$$

hence $V\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) = \frac{2}{3}(S_1 + S_2 + S_3 + S_4) = \frac{2V}{r};$

$$\text{or } \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{r}.$$

ST JOHN'S COLLEGE. Dec. 1836. (No. VII.)

1. In equal circles, angles whether at the centres or circumferences have the same ratio as the arcs which subtend them.

2. Every solid angle is contained by plane angles, which together are less than four right angles.

3. If two pairs of common tangents be drawn to two unequal circles, and 2α, $2\alpha'$ be the angles which the two of each pair make with each other; then

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{R-r}{R+r}.$$

4. A, B, C, D are four points in order in a straight line, find a point E between B and C such that $AE.EB = ED.EC$ by a geometrical construction.

5. If from the centre of a rectangular hyperbola a line be drawn through the point of intersection of two tangents; and if ϕ and ϕ' be the angles which this line and the chord joining the points of contact, respectively make with the real axis; then will $\tan\phi.\tan\phi' = 1$.

6. There are any number of ellipses having a common centre, and their axes majores in the same position. Shew analytically that if all the ellipses be twisted through the same angle θ in the same direction, the loci of the intersections of each ellipse with its original position, are two straight lines whose equations are

$$y = x\tan\frac{\theta}{2}, \text{ and } y = -x\cot\frac{\theta}{2}.$$

7. Two given unequal circles touch each other externally; shew that the locus of the centre of the circle which always touches the other two is a hyperbola. Find the axes and eccentricity, and shew what the figure becomes when the given circles are equal.

8. A vessel whose outward figure is a paraboloid of revolution, is required to be of equal thickness throughout; find the figure of the interior surface.

9. In an ellipse, if through the foci S and H, chords PSP', and QHQ' be drawn parallel to any pair of conjugate diameters, shew that $SP \,.\, SP' + HQ \,.\, HQ' = b^2 + l^2$ where b and l are respectively the semi-axis minor, and semi-latus rectum.

10. In any circle draw a chord AB: from the middle point of the lesser segment draw any line cutting AB in C and meeting the circumference in D; join AD and take $AP = AC$; find the locus of P.

11. Round a given ellipse circumscribe a rhombus; about this rhombus circumscribe a second ellipse, and so on for n times; prove that all the ellipses are similar, and find the sum of the areas of the n ellipses.

12. An ellipse has a square described touching it at the extremities of the minor axis: an ellipse upon the same axis-major circumscribes the square. This ellipse is dealt with in the same manner as before, and the operation is continued till there are altogether $n + 1$ ellipses; prove that if the original eccentricity $= \dfrac{\sqrt{n}}{\sqrt{n+1}}$ the last ellipse becomes a circle.

13. Given the equation $Ay^2 + Bxy + Cx^2 + D = 0$ to be the equation to the hyperbola; find the position of the asymptotes, and the equation to the hyperbola referred to them as axes.

14. Find the axes and position of the curve represented by the equation

$$y^2 - 2xy + 3x^2 + 2y - 4x - 3 = 0.$$

SOLUTIONS TO (No. VII.)

1. Euclid, Prop. 33, Book vi.

2. Euclid, Prop. 11, Book xi.

3. If C, C' (fig. 56) be the centres of the two circles; $DD'T$, $ET'E'$ common tangents to the circles, meeting CC' in T, T' respectively; join CD, $C'D'$, CE, $C'E'$; then if $\angle CTD = \alpha$, $\angle E'T'C' = \alpha'$, we have

$$\sin\alpha = \frac{R-r}{CC'}, \quad \sin\alpha' = \frac{R+r}{CC'};$$

$$\therefore \frac{\sin\alpha}{\sin\alpha'} = \frac{R-r}{R+r}.$$

4. Take any point F (fig. 57) not in AB; about the triangle ABF describe a circle $ABFG$, and about the triangle DCF describe a circle $DCFG$. Let the circles intersect each other in G; join GF and produce it to meet AD in E; then

$$EA \,.\, EB = EF \,.\, EG = EC \,.\, ED.$$

5. Let h, k be the co-ordinates of the point of intersection of the two tangents; then $\tan\phi = \frac{k}{h}$; and the equation to the line joining the points of contact, when tangents are drawn to the hyperbola from the point h, k, is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1;$$

$$\therefore \tan\phi' = \frac{b^2}{a^2}\frac{h}{k}, \quad \text{or} \quad \tan\phi \tan\phi' = \frac{b^2}{a^2};$$

and if the hyperbola be rectangular, $\tan\phi \tan\phi' = 1$.

6. Taking the centre for the origin, the polar equation to the ellipse is $\rho^2 = \frac{b^2}{1 - e^2\cos^2\phi}$; and the polar equation to

the same ellipse when the axis-major is twisted through an angle θ becomes $\rho^2 = \dfrac{b^2}{1 - e^2 \cos^2 (\phi - \theta)}$; therefore at the points of intersection of the ellipses in the two different positions, $\cos^2 \phi = \cos^2 (\phi - \theta)$,

$$\therefore \; \phi = \theta - \phi, \text{ or } \pi + \theta - \phi; \text{ i.e. } \phi = \frac{\theta}{2}, \text{ or } \frac{\pi}{2} + \frac{\theta}{2};$$

both which values are independent of the eccentricity and magnitude of the axes. Hence every corresponding pair of ellipses will intersect each other in two straight lines passing through the origin and inclined at angles $\dfrac{\theta}{2}$ and $\dfrac{\pi}{2} + \dfrac{\theta}{2}$ to the axis-major; or the equations to the two lines will be

$$y = x \tan \frac{\theta}{2}, \quad y = -x \cot \frac{\theta}{2}.$$

7. Let S, H (fig. 58) be the centres of the two circles whose radii are r, r'; P the centre of a circle touching both circles: then $SP - HP = SQ - HR = r - r'$, and is constant, or the locus of P is a hyperbola whose foci are S, H. If $2a$, $2b$ be the axes of the hyperbola,

$$2a = r - r', \quad 2ae = SH = r + r';$$

$$\therefore \; e = \frac{r + r'}{r - r'}, \text{ and } 2b = 2a\sqrt{e^2 - 1} = 2\sqrt{rr'}.$$

When $r = r'$, $SP - HP = 0$, or P lies in the straight line which is drawn perpendicular to SH bisecting it; therefore the locus of P in this case becomes the common tangent.

8. Let θ be the angle which the normal PG (fig. 59) to the parabola makes with the axis AG; X, Y the co-ordinates of P; in PG take $PQ = b$, then the locus of Q will be a curve which by its revolution round AQ will form the inner surface of the vessel.

Since the subnormal $= 2a$, $\tan \theta = \dfrac{Y}{2a}$;

$$\therefore \; Y = 2a \tan \theta, \quad X = a \tan^2 \theta;$$

hence if x, y be the co-ordinates of Q,

$$x = X + b\cos\theta = a\tan^2\theta + b\cos\theta,$$

$$y = Y - b\sin\theta = 2a\tan\theta - b\sin\theta;$$

$$\therefore\ y^2 - 4ax = -\frac{4ab}{\cos\theta} + b^2(1 - \cos^2\theta),$$

$$\text{or}\ \ b^2\cos^3\theta + (y^2 - 4ax - b^2)\cos\theta + 4ab = 0; \quad (1)$$

$$\text{and}\ \ b\cos^3\theta - (x + a)\cos^2\theta + a = 0. \quad (2)$$

Multiply (2) by b and subtract from (1);

$$\therefore\ b(x + a)\cos^2\theta + (y^2 - 4ax - b^2)\cos\theta + 3ab = 0. \quad (3)$$

Multiply (2) by $4b$ and subtract (1),

$$\text{or}\ \ 3b^2\cos^2\theta - 4b(x + a)\cos\theta - (y^2 - 4ax - b^2) = 0. \quad (4)$$

Multiply (3) by $3b$, and (4) by $(x + a)$ and subtract;

$$\therefore\ \{3b(y^2 - 4ax - b^2) + 4b(x + a)^2\}\cos\theta$$
$$+ 9ab^2 + (x + a)(y^2 - 4ax - b^2) = 0.$$

Multiply (3) by $y^2 - 4ax - b^2$, and (4) by $3ab$ and add;

$$\therefore\ \{b(x + a)(y^2 - 4ax - b^2) + 9ab^3\}\cos\theta$$
$$+ (y^2 - 4ax - b^2)^2 - 12ab^2(x + a) = 0.$$

Hence $\qquad \{9ab^2 + (x + a)(y^2 - 4ax - b^2)\}^2$

$$= \{3(y^2 - 4ax - b^2) + 4(x + a)^2\}\{(y^2 - 4ax - b^2)^2 - 12ab^2(x + a)\},$$

which is the equation required.

9. If Aa (fig. 60) be the axis-major, and a' be the semi-diameter parallel to PSP', we have

$$PS\,.\,SP' : SA\,.\,Sa :: a'^2 : a^2;$$

$$\text{or}\quad PS\,.\,SP' = \frac{b^2}{a^2}\,.\,a'^2.$$

$$\text{Similarly,}\quad QH\,.\,HQ' = \frac{b^2}{a^2}\,.\,b'^2;$$

$$\therefore\ PS.SP' + QH\,.\,HQ' = \frac{b^2}{a^2}(a'^2 + b'^2)$$

$$= \frac{b^2}{a^2}(a^2 + b^2) = b^2 + l^2.$$

10. Let E (fig. 61) be the middle point of the circumference AB; join AE, EB; and let

$$\angle EAB = \alpha, \quad \angle EAD = \phi;$$

then $\angle EDA = \angle EBA = \alpha, \quad \angle AED = \pi - (\phi + \alpha),$

$$\angle ECB = \angle EAC + \angle AEC = \pi - \phi; \quad \therefore \angle ACE = \phi;$$

and if $AE = a, \quad AP = AC = \rho, \quad \rho = a\dfrac{\sin(\phi + \alpha)}{\sin\phi};$

the equation to the locus of P.

11. If a', b' be the equal semidiameters of an ellipse, and α the angle between them,

$$2a'^2 = a^2 + b^2, \text{ and } \sin\alpha = \frac{2ab}{a^2 + b^2}; \text{ or } \cos\alpha = \frac{a^2 - b^2}{a^2 + b^2},$$

and the tangents at the extremities of these diameters will form a rhombus whose side $= 2a'$, and whose diagonals are

$$4a' \sin\frac{\alpha}{2}, \text{ and } 4a' \cos\frac{\alpha}{2}.$$

Also the diagonals of a rhombus are at right angles to one another; and if an ellipse be described upon the diagonals as axes it will circumscribe the rhombus; let a_1, b be the semiaxes, then

$$\frac{b_1^2}{a_1^2} = \tan^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{1 + \cos\alpha} = \frac{b^2}{a^2};$$

or the ellipse will be similar to the original ellipse; hence if A, A_1 be the areas of the two ellipses,

$$A_1 = \pi a_1 b_1 = 4\pi a'^2 \sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 2\pi a'^2 \sin\alpha = 2\pi ab = 2A.$$

Similarly, if $A_2, A_3, \ldots A_n$ be the areas of the second, third, and n^{th} ellipses described in the same manner,

$$A_2 = 2A_1 = 2^2 A, \quad A_3 = 2A_2 = 2^3 A, \text{ \&c.}$$

and $A_1 + A_2 + \ldots + A_n = (2 + 2^2 + 2^3 + \ldots + 2^n) A$

$$= (2^{n+1} - 2) A = (2^{n+1} - 2)\pi ab.$$

In this problem it will be necessary to assume, that the ellipses are all described upon the diagonals of the successive rhombuses as axes.

For if $ABEF$ (fig. 62) be a rhombus circumscribing an ellipse, AE, BF its diagonals intersecting one another in C, $AC = \delta$, $BC = \delta'$; then $CP^2 = CD^2 = \dfrac{a^2 + b^2}{2}$,

$$\delta^2 + \delta'^2 = AB^2 = 4CD^2 = 2(a^2 + b^2),$$
$$\delta\delta' = 2\triangle ACB = \tfrac{1}{2}\square ABEF = 2ab;$$

$$\therefore\ \delta = \sqrt{2}\,a,\ \delta' = \sqrt{2}\,.\,b;\ \text{and}\ \frac{x^2}{2a^2} + \frac{mxy}{ab} + \frac{y^2}{2b^2} = 1,$$

will be the equation to an ellipse circumscribing the rhombus, where m may have any magnitude less than unity.

Let e' be the eccentricity of this ellipse; then

$$\frac{2 - e'^2}{e'^2} = \frac{\dfrac{1}{2a^2} + \dfrac{1}{2b^2}}{\sqrt{\left(\dfrac{1}{2b^2} - \dfrac{1}{2a^2}\right)^2 + \left(\dfrac{m}{ab}\right)^2}},$$

$$\text{or}\ \frac{2 - e'^2}{e'^2} = \frac{2 - e^2}{\sqrt{e^4 + 4m^2(1 - e^2)}};$$

hence $\dfrac{2}{e'^2} - 1$ is never greater than $\dfrac{2}{e^2} - 1$; or e' is never less than e, but may have any value greater than e.

12. Let P (fig. 63) be one of the angular points of the square; then if b_1 be the semi-axis minor of the ellipse circumscribing the square, the co-ordinates of P are b, b,

$$\therefore\ \frac{b^2}{a^2} + \frac{b^2}{b_1^2} = 1,\ \text{or}\ \frac{1}{b^2} - \frac{1}{b_1^2} = \frac{1}{a^2};$$

similarly, if $b_2, b_3, \dots b_n$ be the semi-axes minor of the 2^{nd}, $3^{\text{rd}} \dots n^{\text{th}}$ ellipses;

$$\frac{1}{b_1^2} - \frac{1}{b_2^2} = \frac{1}{a^2},$$

$$\frac{1}{b_2^2} - \frac{1}{b_3^2} = \frac{1}{a^2},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{b^2_{n-1}} - \frac{1}{b_n^2} = \frac{1}{a^2};$$

$$\therefore \frac{1}{b^2} - \frac{1}{b_n^2} = \frac{n}{a^2},$$

and if $b_n = a$, the n^{th} ellipse becomes a circle, or $\dfrac{1}{b^2} = \dfrac{n+1}{a^2}$,

$$\text{and } \frac{b^2}{a^2} = 1 - e^2 = \frac{1}{n+1}; \quad \therefore e = \sqrt{\frac{n}{n+1}}.$$

13. $Ay^2 + Bxy + Cx^2 = -D$;

$$\therefore A\left(\frac{y}{x}\right)^2 + B\left(\frac{y}{x}\right) + C = \frac{-D}{x^2};$$

$$\text{hence } \frac{y}{x} + \frac{B}{2A} = \pm\sqrt{\frac{B^2 - 4AC}{4A^2} - \frac{D}{Ax^2}},$$

and the equations to the asymptotes are

$$\frac{y}{x} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

If α, α' be the inclinations of the two asymptotes to the axis of x;

$$\tan\alpha + \tan\alpha' = -\frac{B}{A}; \quad \tan\alpha\tan\alpha' = \frac{C}{A};$$

and if x', y' be the co-ordinates of any point referred to the asymptotes as axes,

$$x = x'\cos\alpha + y'\cos\alpha',$$

$$y = x'\sin\alpha + y'\sin\alpha';$$

$$\therefore (A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha)\,x'^2$$
$$+ (A\sin^2\alpha' + B\sin\alpha'\cos\alpha' + C\cos^2\alpha')\,y'^2$$
$$+ (2A\sin\alpha\sin\alpha' + B\sin(\alpha + \alpha') + 2C\cos\alpha\cos\alpha')\,x'y' + D = 0.$$

Now the two first terms vanish;

$$\therefore\ 2C\cos\alpha\cos\alpha'\left\{\frac{A}{C}\tan\alpha\tan\alpha'+\frac{B}{2C}(\tan\alpha+\tan\alpha')+1\right\}x'y'+D=0,$$

$$\text{or}\ \ 2C\cos\alpha\cos\alpha'\left(2-\frac{B^2}{2AC}\right)x'y'+D=0.$$

$$\text{Now}\ \frac{\sin(\alpha+\alpha')}{\cos\alpha\cos\alpha'}=-\frac{B}{A};\ \ \frac{\cos(\alpha+\alpha')}{\cos\alpha\cos\alpha'}=\frac{A-C}{A};$$

$$\therefore\ \frac{1}{\cos\alpha\cos\alpha'}=\frac{\sqrt{(A-C)^2+B^2}}{A};$$

$$\therefore\ x'y'=\frac{A}{\cos\alpha\cos\alpha'}\ \frac{D}{B^2-4AC}=\frac{\sqrt{(A-C)^2+B^2}}{B^2-4AC}\cdot D.$$

14. The curve is an ellipse since $2^2 < 4 \cdot 1 \cdot 3$.

Let α, β be the co-ordinates of the centre, and let the origin be transferred to the centre by making

$$x = x' + \alpha,\ \ y = y' + \beta;$$

$$\therefore\ (y'+\beta)^2-2(x'+\alpha)(y'+\beta)+3(x'+\alpha)^2+2(y'+\beta)-4(x'+\alpha)-3=0;$$

$$\text{hence}\ \ 2\beta-2\alpha+2=0,\ \ -2\beta+6\alpha-4=0;$$

$$\therefore\ \alpha=\tfrac{1}{2},\ \beta=-\tfrac{1}{2};$$

and the equation to the ellipse becomes

$$y'^2-2x'y'+3x'^2-\frac{9}{2}=0.$$

Let θ be the inclination of the axis of the ellipse to the axis of x, and let $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ be the equation to the curve referred to its principal diameters; transform the axes through an angle θ, then

$$\frac{(x'\cos\theta+y'\sin\theta)^2}{a^2}+\frac{(y'\cos\theta-x'\sin\theta)^2}{b^2}=1;$$

$$\therefore\ \frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{6}{9},$$

$$\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = \frac{2}{9}, \text{ and } \sin 2\theta \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = -\frac{4}{9};$$

$$\text{hence } \frac{1}{a^2} + \frac{1}{b^2} = \frac{8}{9}, \text{ and } \cos 2\theta \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{4}{9},$$

$$\text{or } \tan 2\theta = -1, \text{ and } 2\theta = 135;$$

$$\therefore \frac{1}{b^2} - \frac{1}{a^2} = \frac{4\sqrt{2}}{9}; \quad \frac{1}{b^2} + \frac{1}{a^2} = \frac{8}{9};$$

$$\therefore \frac{2}{a^2} = \frac{8 - 4\sqrt{2}}{9}, \text{ or } a^2 = \frac{9}{4 - 2\sqrt{2}} = \frac{9(2+\sqrt{2})}{4},$$

$$\text{and } b^2 = \frac{9}{4}(2 - \sqrt{2});$$

$$\text{hence } a = 3\cos\frac{\pi}{4}, \; b = 3\sin\frac{\pi}{4}.$$

ST JOHN'S COLLEGE. Dec. 1837. (No. VIII.)

1. If two triangles which have two sides of the one proportional to two sides of the other be joined at one angle so as to have their homologous sides parallel to one another; the remaining sides shall be in a straight line.

2. If a solid angle be contained by three plane angles, any two of them are greater than the third.

3. If from any point in the diagonal of a parallelogram, straight lines be drawn to the angles, the parallelogram will be divided into two pairs of equal triangles.

4. Shew how to find the focus of a traced conic section.

5. From three given centres describe three circles touching one another.

6. SY, HZ are perpendiculars from the foci on the tangent at P to an ellipse whose centre is C; SP, HP cut CY, CZ in Q, R; shew that $CQPR$ is a parallelogram.

7. Let the two circles, radii R, r, which touch first, the three sides of a triangle ABC, and secondly one side BC and the other two produced, touch AB in D_1, D_2, AC in E_1, E_2; shew that $BD_1 . BD_2 = CE_1 . CE_2 = Rr$.

8. The side of an equilateral hexagon inscribed in an ellipse, eccentricity e, with two sides parallel to the axis major : side of one inscribed in the circle on the axis-major :: $4 - 2e^2 : 4 - e^2$.

9. If one of the co-ordinates of the centre of the curve

$ay^2 + bxy + cx^2 + dy + ex + f = 0$, assume the form $\frac{0}{0}$, shew that the equation becomes

$$y = -\frac{bx + d}{2a} \pm \frac{1}{2a}\sqrt{d^2 - 4af},$$

and explain the meaning of it.

10. AP is a parabola, vertex A, focus S; T the point where the axis intersects the directrix; join PT and produce it to meet the latus rectum in N; draw SPQ to meet NQ, which is parallel to ST in Q; and shew that the locus of Q is a circle.

11. If O be a point in the directrix of a parabola; and $OA = a$, $OB = b$, tangents at A and B; shew that the equation to the parabola referred to OA, OB as axes, assumes the form $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

12. Shew that $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, is the equation to a parabola whose latus rectum $= \frac{4a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}$.

13. If in (11) any tangent to the parabola cut OA, OB in P and Q; shew that

$$\frac{OP}{OA} + \frac{OQ}{OB} = 1,$$

where OP or OQ is considered as negative, if P or Q lies in AO or BO produced backwards.

14. Two parallel planes revolve in their own planes about fixed points A, B, in the same direction with equal angular velocities; shew that the curve traced upon the first by a pencil P fixed perpendicular to the plane of the second

is a circle: or if the planes revolve in opposite directions the equation to the curve is

$$a^2 - 2ar\cos\theta + r^2 = \left(\frac{a^2 - r^2}{c}\right)^2$$

where A is the origin, $BP = a$, $AB = c$, and the prime radius is the line originally in the position AB.

15. $ABCD$ is any quadrilateral. Bisect AC, BD in E and F: EF is the locus of the centres of all the inscribed ellipses.

16. Shew that all lines drawn from an external point to touch a sphere are equal to one another; and thence prove that if a tetrahedron can have a sphere inscribed in it, touching its six edges, the sum of every two opposite edges is the same.

SOLUTIONS TO (No. VIII.)

1. Euclid, Prop. 32. Book VI.

2. Euclid, Prop. 20. Book XI.

3. Let $ABCD$ (fig. 64) be the parallelogram whose diagonal is AC; E any point in it; join DE, EB; then since $\triangle ACD = \triangle ABC$, the perpendicular from D on AC equal the perpendicular from B on AC; hence the altitudes of the triangles ADE, ABE are equal; and they are upon the same base, therefore $\triangle ADE = \triangle ABE$. Similarly

$$\triangle DEC = \triangle BEC.$$

4. Find C the centre of the ellipse (fig. 65) by joining the points of bisection of two parallel chords; take any point D in the curve, and with centre C and radius CD describe a circle cutting the ellipse in the four points D, E, F, G; through C draw AA', BB' parallel to DE, EF respectively; these will be the two axes; and with centre B and radius $= AC$ describe a circle cutting AA' in S, H; these will be the two foci required.

5. Let A, B, C (fig. 66) be the three given centres; find O the centre of the circle inscribed in the triangle ABC; draw Oa, Ob, Oc perpendicular to the three sides BC, AC, AB respectively; then $Ab = Ac$, $Cb = Ca$, $Ba = Bc$; and the three circles described with centres A, B, C and radii Ac, Bc, Ca, respectively, will touch one another in the points a, b, c.

6. Produce HP, SY (fig. 67) to meet in V; then since $\angle SPY = \angle YPV$, $SY = YV$, and $HC = CS$; therefore HP is parallel to CY; similarly SP is parallel to CZ; or $CQPR$ is a parallelogram.

7. $BD_1 = S - b$; $BD_2 = S - c$; $CE_1 = S - c$; $CE_2 = S - b$;

$$\therefore\ BD_1 \,.\, BD_2 = (S-b)(S-c) = CE_1 \,.\, CE_2;$$

$$\text{and } R = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}};$$

$$r = \sqrt{\frac{S.(S-b)(S-c)}{S-a}}; \quad \therefore \; Rr = (S-b)(S-c).$$

$$\text{or } BD_1 . BD_2 = CE_1 . CE_2 = Rr.$$

8. Let $APQa$ (fig. 68) be half the hexagon, Aa being one of its diagonals; then if x, y be the co-ordinates of P measured from the centre C,

$$PQ = 2x, \quad AP = 2x;$$

$$\text{and } (a-x)^2 + y^2 = AP^2 = 4x^2,$$

$$\text{or } 3x^2 + 2ax - a^2 = (1-e^2)(a^2 - x^2);$$

$$\therefore \; (4-e^2)x^2 + 2ax = (2-e^2)a^2; \text{ or } x = \frac{2-e^2}{4-e^2}a;$$

and $2x = \dfrac{4-2e^2}{4-e^2}a$; and the side of the hexagon inscribed in the circle on the axis major $= a$; therefore the side of the hexagon inscribed in the ellipse : the side of the hexagon inscribed in the circle on the axis-major :: $4 - 2e^2 : 4 - e^2$.

9. Let α, β be the co-ordinates of the centre; transform the origin to that point by making

$$x = x' + \alpha, \quad y = y' + \beta;$$

$$\therefore \; a(y'+\beta)^2 + b(x'+\alpha)(y'+\beta) + c(x'+\alpha)^2$$
$$+ d(y'+\beta) + e(x'+\alpha) + f = 0;$$

$$\text{hence} \quad 2a\beta + b\alpha + d = 0; \quad b\beta + 2c\alpha + e = 0;$$

$$\therefore \; \alpha = \frac{2ae - bd}{b^2 - 4ac}; \text{ and when } \alpha \text{ assumes the form } \frac{0}{0},$$

$$2ae - bd = 0; \quad b^2 - 4ac = 0.$$

In this case the equation to the curve becomes

$$(2ay + bx)^2 + 2d(2ay + bx) + 4af = 0,$$

$$\therefore \; y = -\left(\frac{bx+d}{2a}\right) \pm \frac{1}{2a}\sqrt{d^2 - 4af} \quad (1)$$

which represents the equations to two parallel straight lines;

and any point in the straight line whose equation is

$$y = -\left(\frac{bx + d}{2a}\right)$$

will be equidistant from the two straight lines represented by equation (1); therefore any line drawn through that point to meet the two lines will be bisected in the same point.

In this case the centre is not limited to a single point; but any point whatever in the straight line $y = -\dfrac{bx + d}{2a}$ will bisect every line passing through it and terminated by the two straight lines represented by the given equation; and will therefore satisfy the definition which has been assumed for the centre.

10. Let AM (fig. 69) $= x$, $MP = y$; draw QN' perpendicular to TS, and let $SN' = x'$, $N'Q = y'$;

$$\text{then } y' = SN = \frac{2a \cdot y}{a + x};$$

$$\text{and } x' = \frac{SM}{MP} \cdot QN' = \frac{a - x}{y} \cdot y';$$

$$\therefore \frac{x'}{y'} + \frac{2a}{y'} = \frac{a - x}{y} + \frac{a + x}{y} = \frac{2a}{y};$$

$$\text{and } \frac{2a}{y'} - \frac{x'}{y'} = \frac{a + x}{y} - \frac{a - x}{y} = \frac{2x}{y} = \frac{4ax}{2ay} = \frac{y}{2a}.$$

Hence by multiplication

$$\frac{4a^2 - x'^2}{y'^2} = 1 \text{ or } x'^2 + y'^2 = 4a^2.$$

The equation to a circle whose centre is S and radius $= 2a = ST$.

11 and 12. See Appendix, 1. Art. 19.

13. Let $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$ be the equation to the parabola; and $\dfrac{x}{m} + \dfrac{y}{n} = 1$, the equation to the tangent; then, at the point in which the tangent meets the curve,

$$y = b\left(1 - 2\sqrt{\frac{x}{a}} + \frac{x}{a}\right) = n\left(1 - \frac{x}{m}\right);$$

and since x has only one value, the quadratic equation

$$\left(\frac{b}{a} + \frac{n}{m}\right)x - \frac{2b}{\sqrt{a}}\sqrt{x} + (b - n) = 0$$

must have its two roots equal;

$$\therefore \frac{b^2}{a} = (b - n)\left(\frac{b}{a} + \frac{n}{m}\right) = \frac{b^2}{a} + \frac{bn}{m} - \frac{bn}{a} - \frac{n^2}{m};$$

$$\text{or } \frac{b}{m} = \frac{b}{a} + \frac{n}{m}; \quad \therefore \frac{m}{a} + \frac{n}{b} = 1.$$

14. First, let the plane of the paper which represents one of the given planes revolve round the point A (fig. 70); and let the other plane revolve round a point whose projection upon the plane of the paper is B. Let the point P describe the angle PBP' round B; then if the plane of the paper revolve in the same direction round A, the straight line AB will move into the position AB', so that

$$\angle BAB' = \angle PBP' = \phi,$$

since the angular velocities are equal; and if

$$AP' = r, \quad \angle P'AB' = \theta,$$

the relation between r and θ will be the polar equation to the curve required. Let $AB = c$, $PB = a$, $\angle ABP = \alpha$, then

$$P'BA = \phi + \alpha,$$

$$\text{and } AP'B = \pi - \{(\phi + \alpha) + (\theta - \phi)\} = \pi - (\theta + \alpha);$$

$$\therefore c^2 = a^2 + r^2 + 2ar\cos(\theta + \alpha),$$

which is the equation to a circle.

Secondly, suppose the planes to move in opposite directions, then AB will move into the position AB_1, and if

$$\angle P'AB_1 = \theta,$$

we have $\angle P'AB = \phi + \theta$, and $\angle P'BA = \phi + \alpha$;

$$\therefore\ r \sin(\phi + \theta) - a \sin(\phi + a) = 0,$$

$$\text{and } r \cos(\phi + \theta) + a \cos(\phi + a) = c\,;$$

$$\text{or } r \cos(\phi + \theta) - a \cos(\phi + a) = c - 2a \cos(\phi + a)\,;$$

and by adding the squares of the two last equations

$$r^2 - 2ar \cos(\theta - a) + a^2 = \{c - 2a \cos(\phi + a)\}^2$$
$$= \left(c - \frac{a^2 + c^2 - r^2}{c}\right)^2 = \left(\frac{a^2 - r^2}{c}\right)^2.$$

15. Let A (fig. 71) be a point without a sphere whose centre is O; AP any straight line drawn from A touching the sphere in P; let the plane APO cut the sphere, the section will be a circle, and AP will be a tangent to this circle;

$$\therefore\ AP^2 = AO^2 - OP^2,$$

and is the same for every position of P.

Next let A (fig. 72) be the vertex, and BDC the base of the tetrahedron; and let the sphere touch AB, AC, DC, DB in the points c', b', b, c respectively; then

$$Ac' = Ab',\ Cb' = Cb,\ Db = Dc,\ Bc = Bc'\,;$$

$$\therefore\ Dc + cB + Ab' + Cb' = Db + Bc' + Ac' + Cb,$$

$$\text{or } BD + AC = AB + CD\,;$$

and in like manner it may be proved that

$$AB + CD = BC + AD.$$

ST JOHN'S COLLEGE. DEC. 1838. (No. IX.)

1. MENTION the principal methods that have been proposed for establishing the theory of parallel straight lines; and shew that the following principle will suffice. Through any point within an angle, a straight line may be supposed to pass which shall cut the two straight lines that contain the angle.

2. Give Euclid's definition of proportion; and apply it to shew that in equal circles, angles at the centres have the same ratio which the circumferences on which they stand have to one another.

3. Every solid angle is contained by plane angles which together are less than four right angles.

4. If from the point where the common tangent to two circles meets the line joining their centres any line be drawn cutting the circles, it will cut off similar segments.

5. Of the two squares which can be inscribed in a right-angled triangle, which is the greatest?

6. Construct all right-angled triangles whose sides shall be rational, upon a given straight line as their base.

7. In the sides AB, AC of a given triangle ABC, take two points M, N; and in the line joining them, take a point P such that $\frac{MB}{AM} = \frac{AN}{NC} = \frac{MP}{PN}$; prove that if PB, PC be joined, the triangle PBC is twice the triangle AMN.

8. In No. 7 the circle described about the triangle AMN will always pass through a fixed point.

9. Draw the straight lines represented by the equation

$$(2y - x + c)(3y + x - c) = 0;$$

and determine where they intersect, and at what angle.

10. Find the equation to the line which is equidistant from the two lines represented by the equation

$$y = mx + c \pm c'.$$

11. The circles represented by the equation

$$(n + 1)(x^2 + y^2) = ax + bny,$$

when n assumes various values, will have a common chord.

12. Find the locus of a point at which the base of a triangle subtends an angle equal to the sum of the angles at the base.

13. In No. 7 find the locus of P.

14. If the vertex and nearer focus of an ellipse remain fixed, whilst the centre moves in the line joining them to an infinite distance, the curve will become a parabola; but if it move in the opposite direction, the curve will become successively a circle, a straight line, hyperbola, and parabola.

15. An ellipse and hyperbola that have the same foci and centre will cut one another at right angles.

16. In No. 15 if from any point in the circumference of the circle which passes through the points of intersection of the ellipse and hyperbola, tangents be drawn to those curves, they will be at right angles.

17. Trace the curve whose equation is

$$y = \tfrac{1}{4}x + 2 \pm \sqrt{-x^2 - 3x + 10}.$$

18. If the length of the axis of an oblique cone be equal to the radius of the base, every section perpendicular to the axis will be a circle.

19. In the general equation of the second order

$$ay^2 + bxy + cx^2 + dy + ex + f = 0;$$

shew what the curve which it represents becomes when

$$b^2 - 4ac = \pm \infty.$$

20. Find the conditions in order that two given equations of the second order may represent similar and similarly situated curves.

SOLUTIONS TO (No. IX.)

1. See Potts' Euclid, p. 50.

2. Euclid, Def. 5, Book v. and Prop. 33, Book vi.

3. Euclid, Prop. 21, Book xi.

4. Let C (fig. 73) be the point in which the common tangent CDE meets the straight line CBA joining the centres of the two circles; $CFGHK$ a straight line cutting both circles; join BG, BD, BF, DG, DF, AK, AE, AH, EK, EH; then because the angles CDB, CEA are right angles, the triangles CDB, CEA are similar;

$\therefore\ CD : CE :: CB : CA :: BD : EA :: BF : AH :: CF : CH$;

hence the triangle CDF, CEH are similar, and DF is parallel to EH. Similarly, DG is parallel to EK, $\therefore\ \angle FDG = \angle KEH$, and the segments FDG, HEK are similar.

5. Let x be the side of the square $abcd$ (fig. 74) inscribed in the triangle ABC, and having one of its sides cd on the hypothenuse AB; then

$Ca = x \sin A$, $Ba = x \sec A$, or $x(\sin A + \sec A) = BC = a$;

$$\therefore\ x = \frac{a \cos A}{1 + \sin A \cos A}.$$

Next, let x' be the side of the square $Ca'c'b'$ (fig. 75) which has two of its sides coincident with the two sides CA, CB of the triangle; then

$$CB = Ba' + Ca', \text{ or } x' \tan A + x' = a;$$

$$\therefore\ x' = \frac{a \cos A}{\sin A + \cos A}.$$

Now $(1 - \sin A)(1 - \cos A)$ is positive;

$$\therefore\ 1 - \sin A - \cos A + \sin A \cos A > 0,$$

$$\text{or } 1 + \sin A \cos A > \sin A + \cos A;$$

hence x is less than x'; and the square which has one of its angular points in the hypothenuse is the greatest.

6. Let x and mx be the two sides; then

$$x^2(1+m^2) = (\text{hypothenuse})^2 = x^2\left(m+\frac{1}{n}\right)^2;$$

$$\text{or } 1+m^2 = m^2 + \frac{2m}{n} + \frac{1}{n^2}; \quad \therefore m = \frac{n^2-1}{2n};$$

and if $x = 2na$, the sides of the triangle are $2na$, $(n^2-1)a$; and the hypothenuse $= (n^2+1)a$.

Hence if AB (fig. 76) be the base, divide AB into $2n$ equal parts, and let AD, DB contain $n+1$ and $n-1$ parts respectively, and BE one part; then if BC be a fourth proportional between BE, BD and AD, $BC = (n^2-1)a$, and $AC = (n^2+1)a$; or ABC will be the triangle required.

7. Let AM (fig. 77) $= a$, $MB = na$; $\therefore AB = (n+1)a$;

$$NC = b, \quad AN = nb, \quad AC = (n+1)b;$$

$$PN = c, \quad PM = nc, \quad MN = (n+1)c;$$

$$\therefore \triangle ABC = \tfrac{1}{2}AB.AC.\sin A = \tfrac{1}{2}(n+1)^2 ab \sin A;$$

$$\triangle AMN = \tfrac{1}{2}AM.AN\sin A = \tfrac{1}{2}n\,ab\sin A,$$

$$\text{or } \triangle ABC = \frac{(n+1)^2}{n}\triangle AMN;$$

$$\text{and } \triangle PNC = \frac{PN.NC}{MN.AN}\triangle ANM = \frac{1}{n.(n+1)}\triangle AMN;$$

$$\triangle PMB = \frac{PM.MB}{NM.MA}\triangle AMN = \frac{n^2}{n+1}\triangle AMN;$$

$$\therefore \triangle PBC = \triangle ABC - \triangle AMN - \triangle PNC - \triangle PMB$$

$$= \left\{\frac{(n+1)^2}{n} - 1 - \frac{1}{n(n+1)} - \frac{n^2}{n+1}\right\}\triangle AMN = 2\triangle AMN.$$

8. Let $AB = \alpha$, $AC = \beta$; $\therefore AM = \dfrac{\alpha}{n+1}$, $AN = \dfrac{n\beta}{n+1}$,

and the equation to the circle passing through A, M, N, referred to AB, AC as axes becomes

$$x^2 + y^2 - 2xy\cos A = a_1 x + b_1 y,$$

but when $y = 0$, $x = AM = a_1$; $\therefore\ a_1 = \dfrac{a}{n+1}$;

similarly, when $x = 0$, $y = b_1 = AN$, or $b_1 = \dfrac{n\beta}{n+1}$,

and the equation to the circle is

$$x^2 + y^2 - 2xy\cos A = \frac{ax + n\beta y}{n+1}.$$

Hence if $ax = \beta y$; $x^2 + y^2 - 2xy\cos A = ax = \beta y$; from which equations two values of x and y may be found independent of n; therefore all the circles will pass through the two points so determined; hence they have a common chord whose equation is $ax = \beta y$; and $x = \dfrac{a\beta^2}{(BC)^2}$; $y = \dfrac{\beta a^2}{(BC)^2}$ are the co-ordinates of the common point through which they all pass.

9. The equations to the two straight lines are

$$2y - x + c = 0, \text{ and } 3y + x - c = 0;$$

and by addition $5y = 0$, or $y = 0$, and $x = c$; hence the two straight lines intersect in the axis of x at a distance c from the origin.

Let m, m' be the tangents of the angles which the two straight lines make with the axis of x; and θ the angle between them; $\therefore\ m = \frac{1}{2}$, $m' = -\frac{1}{3}$,

$$\text{and } \tan\theta = \frac{m' - m}{1 + mm'} = \frac{-\frac{1}{3} - \frac{1}{2}}{1 - \frac{1}{6}} = -1;\ \theta = 135^\circ.$$

10. Let
$$x\cos a + y\sin a = p, \quad (1)$$
$$x\cos a + y\sin a = p', \quad (2)$$
$$x\cos a + y\sin a = p'', \quad (3)$$

be the equations to three parallel straight lines, whose perpendicular distances from the origin are p, p', p'' respectively; then if the second straight line be equidistant from the first and third,

$$p' = \frac{p + p''}{2};$$

and if $\cot a = -m$, the equations are

$$y = mx + \frac{p}{\sin a}, \quad y = mx + \frac{p'}{\sin a}, \quad y = mx + \frac{p''}{\sin a};$$

or the second equation is

$$y = mx + \tfrac{1}{2}\left(\frac{p}{\sin a} + \frac{p''}{\sin a}\right).$$

Hence in the proposed example the equation becomes,

$$y = mx + \tfrac{1}{2}\left\{(c + c') + (c - c')\right\} = mx + c.$$

11. Equating the coefficients of n, we have

$$x^2 + y^2 = by, \text{ and } x^2 + y^2 = ax;$$

hence the straight line whose equation is $ax = by$ will meet all the circles in the same point, or the circles have a common chord whose equation is $ax = by$.

12. Let C be the vertical angle, and

$$\angle C = \angle A + \angle B = \pi - C;$$

$\therefore \angle C = 90^0$, and the locus of C is a circle described on the diameter AB.

13. Draw PV (fig. 77) parallel to AC meeting AB in V; and let AB, AC be taken for the axes of x and y respectively; then if $AV = x$, $VP = y$,

$$AM = (n+1)\,x; \quad AN = \frac{(n+1)}{n}\,y;$$

$$AB = (n+1)\,AM = (n+1)^2\,x = a;$$

$$AC = \frac{n+1}{n}\,AN = \left(\frac{n+1}{n}\right)^2 y = b;$$

$$\therefore \sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = \frac{1}{n+1} + \frac{n}{n+1} = 1;$$

which is the equation to a parabola referred to two tangents AB, AC.

14. Let A be the vertex, S the focus,

$$AS = c = a(1 - e); \quad \therefore \; 1 + e = 2 - \frac{c}{a},$$

and
$$\begin{aligned} y^2 &= (1 - e^2)(2ax - x^2) \\ &= (1 + e)\{2a(1 - e)x - (1 - e)x^2\} \\ &= \left(2 - \frac{c}{a}\right)\left(2cx - \frac{c}{a}x^2\right). \end{aligned}$$

If $a > c$, $2 - \dfrac{c}{a}$ is positive; and as a increases the curve continues to be an ellipse, having the axis of x for its axis-major until $a = \infty$, when $y^2 = 4cx$, and the curve approaches to a parabola.

If a diminishes until $a = c$; the equation becomes

$$y^2 = 2cx - x^2,$$

or the curve becomes a circle when C moves up to S.

If $a < c > \dfrac{c}{2}$, the curve becomes an ellipse having its axis-major in the direction of the axis of x.

When $a = \dfrac{c}{2}$, $y = 0$, and the ellipse coincides with the axis of x.

When $a < \dfrac{c}{2}$, $y^2 = \left(\dfrac{c}{a} - 2\right)\left(\dfrac{c}{a}x^2 - 2cx\right)$, and the curve becomes a hyperbola.

When $a = 0$, $y^2 = \left(\dfrac{c}{a}\right)^2 x^2$, or $x = 0$, and the curve coincides with the axis of y. (This result is not mentioned in the problem.)

When a is negative, $y^2 = \left(\dfrac{c}{a} + 2\right)\left(\dfrac{c}{a}x^2 + 2cx\right)$, which is still the equation to a hyperbola.

When $a = -\infty$, $y^2 = 4cx$, and the curve again becomes a parabola.

15. Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and $\dfrac{x^2}{a'^2} - \dfrac{y^2}{b'^2} = 1$ be the equations to the ellipse and hyperbola; then $CS^2 = a^2 - b^2 = a'^2 + b'^2$, since the curves have the same foci and centre; and at the points of intersection

$$\left(\frac{1}{a^2} - \frac{1}{a'^2}\right) x^2 + y^2 \left(\frac{1}{b^2} + \frac{1}{b'^2}\right) = 0;$$

$$\text{or}\quad \frac{x^2}{y^2} = \frac{a^2 a'^2}{b^2 b'^2} \left(\frac{b^2 + b'^2}{a^2 - a'^2}\right) = \frac{a^2 a'^2}{b^2 b'^2}.$$

Now if θ, θ' be the angles which the tangents to the two curves at the points of intersection make with the axis of x,

$$\tan\theta = -\frac{b^2 x}{a^2 y}, \quad \tan\theta' = \frac{b'^2 x}{a'^2 y};$$

$$\therefore \tan\theta \tan\theta' = -\frac{b^2 b'^2}{a^2 a'^2} \frac{x^2}{y^2} = -1;$$

or the tangents at the points of intersection of the two curves are at right angles.

16. Let a tangent be drawn to an ellipse from a point h, k, and let m be the tangent of the angle which the tangent makes with the axis of x; then its equation is

$$y - mx = \sqrt{b^2 + a^2 m^2};$$

and since it passes through the point h, k,

$$k - mh = \sqrt{b^2 + a^2 m^2}.$$

Similarly, if a tangent be drawn from the point h, k to an hyperbola whose semiaxes are a', b', and m_1 be the tangent of its inclination to the axis of x,

$$k - m_1 h = \sqrt{a'^2 m_1^2 - b'^2};$$

and when the tangents to the ellipse and hyperbola are at right angles

$$m_1 = -\frac{1}{m}, \text{ or } km + h = \sqrt{a'^2 - b'^2 m^2};$$

$$\text{and } k - mh = \sqrt{b^2 + a^2 m^2};$$

therefore adding the squares of these two equations,

$$(1 + m^2)(h^2 + k^2) = b^2 + a'^2 + (a^2 - b'^2) m^2 = (1 + m^2)(b^2 + a'^2);$$

$$\text{hence } h^2 + k^2 = b^2 + a'^2. \quad (1)$$

But if x, y be the co-ordinates of the point of intersection of the ellipse and hyperbola,

$$\frac{x^2}{y^2} = \frac{a^2 a'^2}{b^2 b'^2}; \quad \therefore \frac{x^2}{a^2} = \frac{a'^2}{b'^2}\frac{y^2}{b^2}, \text{ or } \frac{y^2}{b^2}\left(1 + \frac{a'^2}{b'^2}\right) = 1;$$

$$\text{hence } y^2 = \frac{b^2 b'^2}{a'^2 + b'^2}, \quad x^2 = \frac{a^2 a'^2}{a'^2 + b'^2};$$

$$\text{and } x^2 + y^2 = \frac{a^2 a'^2 + b^2 b'^2}{a'^2 + b'^2} = \frac{a'^2 (a^2 - b^2) + b^2 (b'^2 + a'^2)}{a'^2 + b'^2}$$

$$= \frac{a'^2 (a'^2 + b'^2) + b^2 (a'^2 + b'^2)}{a'^2 + b'^2} = a'^2 + b^2,$$

which is the equation to a circle passing through the points of intersection of the ellipse and hyperbola: and from equation (1), h, k are co-ordinates of any point in this circle.

17. $y = \frac{1}{4}x + 2 \pm \sqrt{(2 - x)(5 + x)}$. Let A (fig. 78) be the origin; draw the straight line BCD whose equation is $y = \frac{1}{4}x + 2$ by making $AB = 8$, $AC = 2$; then BCD is a diameter to the curve; and if $x = 2$ or $x = -5$, the curve will intersect the diameter in the points D, E. Bisect ED in G, then G will be the centre, and the curve will be symmetrical with respect to ED, having equal portions above and below ED. When $x = 0$, $y = 2 \pm \sqrt{10}$;

$$\text{hence if } AH = 2 + \sqrt{10}, \text{ and } AF = \sqrt{10} - 2,$$

the curve will pass through the points F, H, and it is manifest that x cannot be greater than 2 nor less than -5, hence the abscissæ of D and E are the greatest possible, and the tangents at those points will be parallel to the axis of y.

18. Let AD (fig. 79) be the axis; BAC a section of

the cone made by a plane through the axis perpendicular to the base; then since

$$AD = DB, \quad \angle BAD = \angle ABD = \alpha;$$

$$\text{also } AD = DC; \quad \therefore \angle DAC = \angle DCA = \beta;$$

$$\text{hence } \angle BAC = \alpha + \beta; \quad \text{and } 2(\alpha + \beta) = \pi,$$

$$\text{or } \alpha + \beta = \frac{\pi}{2};$$

but the angle AED which the circular section of the cone makes with $AB = \angle ACB = \beta$;

$$\therefore \angle ADF = \angle AED + \angle EAD = \beta + \alpha = \frac{\pi}{2};$$

or every section perpendicular to the axis will be circular.

19. When $b^2 - 4ac = \infty$, we must have one or more of the quantities $a, b, c = \infty$.

(1) Let a be infinite, and b, c finite; then $y^2 = 0$, and the equation represents the axis of x.

(2) Let b be infinite, and a, c finite; then

$$xy = 0; \quad \therefore x = 0, \text{ or } y = 0;$$

and the equation represents the axes of y and x.

(3) Let c be infinite and a, b, finite; then $x^2 = 0$, and the equation represents the axis of y.

(4) Let a, b, c be infinite, and d, e, f finite;

$$\text{then } y^2 + \frac{b}{a}xy + \frac{c}{a}x^2 = 0,$$

$$\text{or } \frac{y}{x} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \left(\frac{c}{a}\right)} \quad (1)$$

which may be finite or infinite, depending upon the values of $\frac{b}{a}$, and $\frac{c}{a}$.

If $b^2 - 4ac$ be negative, equation (1) can only be satisfied by supposing

$$\frac{b^2 - 4ac}{a^2} = 0, \text{ or } \frac{b}{a} = 2\sqrt{\frac{c}{a}},$$

in which case $\dfrac{y}{x} = -\dfrac{b}{2a}$ which is the equation to a straight line.

Hence when $b^2 - 4ac = \pm \infty$, the equation represents one or two straight lines.

20. Let $ay^2 + bxy + cx^2 + dy + ex + f = 0$;

$$a'y'^2 + b'x'y' + c'x'^2 + d'y' + e'x' + f' = 0,$$

be the equations to two curves similar and similarly situated; then if $x' = mx$, y' must $= my$;

$$\therefore\ a'm^2y^2 + b'm^2xy + c'm^2x^2 + d'my + e'mx + f' = 0$$

must agree with the equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0;$$

$$\therefore\ \frac{a}{f} = m^2\frac{a'}{f'};\ \frac{b}{f} = m^2\frac{b'}{f'},\ \frac{c}{f} = m^2\frac{c'}{f'};$$

$$\frac{d}{f} = m\frac{d'}{f'},\ \frac{e}{f} = m\frac{e'}{f'}.$$

$$\text{or } m^2 = \frac{af'}{a'f} = \frac{bf'}{b'f} = \frac{cf'}{c'f} = \frac{d^2f'^2}{d'^2f^2} = \frac{e^2f'^2}{e'^2f^2};$$

$$\therefore\ \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d^2}{d'^2}\cdot\frac{f'}{f} = \frac{e^2}{e'^2}\cdot\frac{f'}{f},$$

which are the conditions required.

ST JOHN'S COLLEGE. Dec. 1839. (No. X.)

1. Describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

2. Explain and illustrate the fifth and seventh definitions in the fifth book of Euclid; and shew that a magnitude has a greater ratio to the less of two unequal magnitudes than it has to the greater.

3. O is the centre of the circle inscribed in an equilateral triangle ABC; shew that if AD be drawn perpendicular to the base intersecting the circle in E, AD is divided into three equal parts in O and E.

4. ABC is an equilateral triangle; AF, BE are drawn perpendicular to the sides BC, AC intersecting each other in D; shew that if FG be drawn to the middle point of AB, it will be a tangent to the circle described about $CEDF$.

5. If in the figure of Euclid, Book IV. Prop. 10 the straight lines DC, BA be produced to meet the circle again in E, F, and EF be joined, shew that the triangle CEF is to the triangle ABD as $3 + \sqrt{5} : 2$, and that the triangle ABD is a mean proportional between CEF and BCD.

6. O is a point within the triangle ABC: D, E, F any points in the sides BC, AC, AB respectively; shew that if OD, OE, OF be joined, and Aa, Bb, Cc be drawn parallel to them from the angles A, B, C to the opposite sides, then

$$\frac{OD}{Aa} + \frac{OE}{Bb} + \frac{OF}{Cc} = 1.$$

7. Find the polar co-ordinates of the points of intersection of the straight lines

$$\rho = 2a \sec\left(\theta - \frac{\pi}{2}\right), \quad \rho = a \sec\left(\theta - \frac{\pi}{6}\right),$$

and the angle between them.

8. Prove *analytically* that the angles in the same segment of a circle are equal to each other.

9. Lp is a normal to a parabola at L the extremity of the latus rectum, meeting the parabola again in p; shew that the diameter in which the tangents at L and p intersect, passes through l the other extremity of the latus rectum.

10. Shew that if two equal parabolas have their axes in the same straight line and towards the same parts, the segment of the exterior one cut off by any straight line which touches the interior is invariable so long as the distance between the vertices is unchanged.

11. If SQ be drawn always bisecting the angle PSC in an ellipse, and equal to the mean proportional between SC and SP, find the eccentricity of the curve which is the locus of Q.

12. $CPLD$ is a parallelogram whose sides CP, CD are semiconjugate diameters of a rectangular hyperbola inclined to one another at an angle of 60^0, find the equation to the ellipse which passes through C, P, L, D, and cuts the conjugate hyperbola at D at an angle of 15^0.

13. Define similar curves; and shew that all curves similar to that whose equation referred to rectangular axes is $y = Fx$ are included in the equation

$$\begin{aligned}&k(y-b)\cos\theta - k(x-a)\sin\theta\\ &= F\{k(y-b)\sin\theta + k(x-a)\cos\theta\}.\end{aligned}$$

14. CP, CD are any semiconjugate diameters of an ellipse; join DP, draw CP' parallel to DP, and join PP'; then the area of the trapezium $CP'PD$ is to that of the ellipse as $1 + \dfrac{1}{\sqrt{2}} : 2\pi$.

15. SY, HZ are perpendiculars from the foci of an ellipse upon the tangent at any point; find the locus of the point in which HY, SZ intersect each other.

16. AB, AG, AD are three edges of a given parallelopiped inclined to each other at angles α, β, γ respectively, and to the vertical at angles θ, ϕ, ψ respectively; in the side $BECF$ a point P is taken whose co-ordinates referred to the oblique axes BE, BF are h, k; find the inclination of the straight line AP to the horizon.

17. Three circles are so inscribed in a triangle that each touches the other two and two sides of the triangle; prove that the radius of that which touches the sides AB, AC is

$$\frac{r}{2}\left\{\frac{\left(1+\tan\frac{B}{4}\right)\left(1+\tan\frac{C}{4}\right)}{1+\tan\frac{A}{4}}\right\},$$

r being the radius of the circle inscribed in the triangle.

SOLUTIONS TO (No. X.)

1. Euclid, Prop. 25, Book vi.

2. See Potts' Euclid, p. 162, and Euclid, Prop. 8. Book v.

3. (Fig. 80) $AF = \dfrac{AC}{2} = DC$: and

$$AD^2 = AC^2 - CD^2 = 4CD^2 - CD^2 = 3CD^2 = 3AF^2 = 3AE \,.\, AD;$$

$$\therefore\ AD = 3AE, \text{ or } AE = \frac{AD}{3};$$

$$\text{also } ED = AD - AE = \frac{2AD}{3};$$

$$\therefore\ OE = \frac{AD}{3}, \text{ and } OD = \frac{AD}{3}.$$

4. Since the angles at E and F (fig. 81) are right angles, a circle may be described round $CEDF$; also since AB, BC are bisected in G and F, FG is parallel to AC,

$$\text{and } \angle BFG = 60^0;\ \therefore\ \angle AFG = 30^0 = \angle DCF;$$

hence FG is a tangent to the circle.

5. Let $AB = a$;

$$\therefore\ AC = BD = 2a \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{2}\,a = CD;$$

$$\text{also } CF = a + Ac = \frac{\sqrt{5}+1}{2}\,a;$$

$$\therefore\ AC \,.\, CF = a^2;$$

$$\text{and } BC = a - AC = a\left(\frac{3-\sqrt{5}}{2}\right) = a\left(\frac{\sqrt{5}-1}{2}\right)^2;$$

$$\text{and } CE \,.\, CD = BC \,.\, CF = \left(\frac{\sqrt{5}-1}{2}\right) a^2;$$

$$\therefore\ CE = a;$$

$$\text{hence} \quad \triangle ECF = \tfrac{1}{2} . CE . CF . \sin ECF,$$
$$\triangle ABD = \tfrac{1}{2} AB . BD . \sin ABD,$$
$$\triangle BCD = \tfrac{1}{2} BC . CD . \sin BCD;$$
$$\therefore \frac{\triangle ECF}{\triangle ABD} = \frac{CF}{BD} = \frac{CF}{CD} = \frac{CE}{BC} = \frac{AB}{BC} = \frac{\triangle ABD}{\triangle BCD};$$
$$\text{hence} \quad (\triangle ECF)(\triangle BCD) = (\triangle ABD)^2,$$
$$\text{and} \quad \triangle ECF : \triangle ABD :: AB : BC :: 3 + \sqrt{5} : 2.$$

6. (Fig. 82) $\triangle BOC = \dfrac{OD}{Aa} \triangle ABC,$

$$\triangle AOC = \frac{OE}{Bb} \triangle ABC, \quad \triangle AOB = \frac{OF}{Cc} \triangle ABC;$$

therefore by addition

$$\triangle ABC = \left(\frac{OD}{Aa} + \frac{OE}{Bb} + \frac{OF}{Cc}\right) \triangle ABC;$$
$$\text{or} \quad \frac{OD}{Aa} + \frac{OE}{Bb} + \frac{OF}{Cc} = 1.$$

7. If p be the perpendicular upon the straight line from the origin, and α the angle which the perpendicular makes with the first radius, the equation to the straight line is

$$\rho = p \sec (\theta - \alpha);$$

and if $\rho = p' \sec (\theta - \alpha')$ be the equation to another straight line, the angle between them equal the angle between the perpendiculars $= \alpha - \alpha'$; hence the angle between the two given straight lines $= \dfrac{\pi}{2} - \dfrac{\pi}{6} = \dfrac{\pi}{3}$.

At the point of intersection

$$2a \sec \left(\theta - \frac{\pi}{2}\right) = a \sec \left(\theta - \frac{\pi}{6}\right);$$
$$\therefore 2 \cos \left(\theta - \frac{\pi}{6}\right) = \sin \theta; \quad \text{or} \quad \theta = \frac{\pi}{2}:$$
$$\text{and} \quad \rho = 2a \sec \left(\theta - \frac{\pi}{2}\right) = 2a.$$

8. Let A (fig. 83) be the origin, AB the axis of x; and the equation to the circle $x^2 + y^2 = ax + by$; therefore when $y = 0$, $x = AB = a$; and if x', y' be the co-ordinates of any point P,

$$\tan PAB = \frac{y'}{x'}, \quad \tan PBA = \frac{y'}{a - x'};$$

$$\therefore \tan APB = -\tan(PAB + PBA)$$

$$= -\frac{\dfrac{y'}{x'} + \dfrac{y'}{a - x'}}{1 - \dfrac{y'^2}{ax' - x'^2}} = \frac{ay'}{x'^2 + y'^2 - ax'} = \frac{ay'}{by'} = \frac{a}{b};$$

hence $\angle APB = \tan^{-1}\dfrac{a}{b}$, and is independent of the position of P.

9. Generally let QSq (fig. 84) be any chord of a parabola passing through S, Qp a normal at Q; then since the tangents at Q and q are at right angles, Qp is parallel to the tangent at q; therefore the diameter to Qp passes through q; but the tangents at Q and p intersect in the diameter to Qp; hence the diameter in which the tangents at Q and p intersect will pass through q.

If Q be one extremity of the latus rectum, q will be the other extremity.

10. Let A, a (fig. 85) be the vertices of the two parabola; QPq a tangent to the inner parabola; draw Pp parallel to the axis meeting the exterior parabola in p; and draw PM, pm perpendicular to the axis; then if L be the latus rectum,

$$AM = \frac{PM^2}{L} = \frac{pm^2}{L} = am;$$

therefore $Pp = Aa$; and if α be the angle which Qq makes with the axis,

$$PQ^2 = \frac{L}{\sin^2 \alpha} Pp;$$

hence area of segment $Qpq = \frac{4}{3} Pp \,.\, PQ \,.\, \sin \alpha$

$$= \tfrac{4}{3} Pp \sqrt{L \,.\, Pp} = \tfrac{4}{3} \sqrt{L (Aa)^3},$$

and is constant.

11. Let $SQ = \rho$; (fig. 86) $\angle CSQ = \theta$;

$$\therefore \angle PSC = 2\theta, \text{ and } SP = \frac{a(1 - e^2)}{1 - e \cos 2\theta};$$

$$\therefore \rho^2 = (ae)(SP) = \frac{a^2 e (1 - e^2)}{1 - e \cos 2\theta};$$

$$\text{or } \rho^2 \left\{1 - e(\cos^2 \theta - \sin^2 \theta)\right\} = a^2 e (1 - e^2);$$

$$\therefore x^2 + y^2 - e(x^2 - y^2) = a^2 e (1 - e^2),$$

$$\text{or } (1 - e) x^2 + (1 + e) y^2 = a^2 e (1 - e^2);$$

$$\text{hence } \frac{x^2}{a^2 e (1 + e)} + \frac{y^2}{a^2 e (1 - e)} = 1;$$

which is the equation to an ellipse whose centre is S, semiaxes $a\sqrt{e(1 + e)}$, and $a\sqrt{e(1 - e)}$, and eccentricity $\sqrt{\dfrac{2e}{1 + e}}$.

12. Since the hyperbola is rectangular, $CP = CD$ (fig. 87); and if CL, DP be joined to meet in E, $\angle DEL$ is a right angle; and E will be the centre of the ellipse which passes through the points C, P, L, D.

Let CA be the semiaxis of the hyperbola $= a$;

$$CP = a'; \quad \therefore a'^2 \sin 60 = a^2, \text{ or } a'^2 \frac{\sqrt{3}}{2} = a^2;$$

$$DE = \frac{a'}{2}; \quad EL = a' \frac{\sqrt{3}}{2}.$$

and the equation to the ellipse is

$$\frac{x^2}{EL^2} + \frac{y^2}{ED^2} + 2m \frac{xy}{a'} = 1,$$

where m is a constant to be determined;

$$\therefore \frac{4x^2}{3a'^2} + \frac{4y^2}{a'^2} + 2m \frac{xy}{a'} = 1;$$

and since the tangent at D makes an angle of 15^0 with DL, its equation is

$$\frac{x}{a'\left(1+\frac{\sqrt{3}}{2}\right)}+\frac{2y}{a'}=1,$$

$$\text{or } \frac{2y}{a'}=1-\frac{(4-2\sqrt{3})\,x}{a'};$$

$$\therefore\ \frac{4x^2}{3a'^2}+\left\{1-\left(\frac{4-2\sqrt{3}}{a'}\right)x\right\}^2+mx\left(1-\frac{4-2\sqrt{3}}{a'}x\right)=1;$$

and since in this equation x must have only one value, the coefficient of x will vanish;

$$\therefore\ m-2\frac{(4-2\sqrt{3})}{a'}=0,$$

and the equation becomes

$$\frac{4x^2}{3a'^2}+\frac{4y^2}{a'^2}+\frac{4(4-2\sqrt{3})}{a'^2}xy=1.$$

13. Similar curves are such that if two lines can be taken for the abscissæ, and any two lines equally inclined to them for the ordinates, if the abscissæ be taken in a constant ratio, the corresponding ordinates will always be in the same ratio.

If $y=Fx$ be the equation to a curve; let the origin be changed to a point whose co-ordinates are $-\alpha$, $-\beta$, and the axes transformed through an angle θ; then the equation becomes

$$(y-\beta)\cos\theta-(x-\alpha)\sin\theta=F\left\{(y-\beta)\sin\theta+(x-\alpha)\cos\theta\right\};$$

and the equation to a similar curve is found by putting kx and ky for x and y respectively; and if $\alpha=ka$, $\beta=kb$, the equation becomes

$$k(y-b)\cos\theta-k(x-a)\sin\theta=F\left\{k(y-b)\sin\theta+k(x-a)\cos\theta\right\}.$$

14. Let the ellipse be referred to two conjugate diameters CP, CD; (fig. 88) then the equation to PD is $\frac{x}{a'}+\frac{y}{b'}=1$;

and the equation to CP' is $\frac{x}{a'} + \frac{y}{b'} = 0$; hence for the co-ordinates of P',

$$\frac{y^2}{b'^2} + \frac{y^2}{b'^2} = 1\,;\ \text{ or } y = \frac{b'}{\sqrt{2}}\,;$$

$$\text{and } \triangle PCP' = \tfrac{1}{2}\, CP \,.\, y \sin\alpha = \tfrac{1}{2}\,\frac{a'b'\sin\alpha}{\sqrt{2}} = \frac{ab}{2\sqrt{2}}\,;$$

$$\text{also } \triangle PCD = \tfrac{1}{2}\, a'b' \sin\alpha = \frac{ab}{2}\,;$$

$$\therefore \text{ trapezium } CDPP' = \frac{ab}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \left(\frac{1 + \frac{1}{\sqrt{2}}}{2\pi}\right)(\pi ab).$$

15. Draw the normal PG (fig. 89); then

$$\frac{SG}{GH} = \frac{SP}{PH} = \frac{PY}{PZ} = \frac{SY}{HZ} = \frac{SQ}{QZ}\,;$$

therefore Q the point of intersection of SZ, HY is in the normal PG.

$$\text{Also } \frac{PQ}{HZ} = \frac{YP}{YZ} = \frac{SG}{SH} = \frac{QG}{HZ}\,;$$

$\therefore$ $PQ = QG$; and PG is bisected in the point Q.

Hence if x, y be the co-ordinates of P; x', y' the co-ordinates of Q;

$$x' = CG + \tfrac{1}{2}\, GM = e^2 x + \tfrac{1}{2}(1 - e^2)\, x = \frac{1 + e^2}{2}\, x\,;$$

$$y' = \frac{PM}{2} = \frac{y}{2}\,;\quad \therefore\ x = \frac{2}{1 + e^2}\, x'\,;\ y = 2y'\,;$$

$$\text{or } \left\{\frac{2x'}{a(1 + e^2)}\right\}^2 + \frac{4y'^2}{b^2} = 1\,;$$

which is the equation to an ellipse whose centre is C, and axes $a(1 + e^2)$ and b respectively.

16. Let AP (fig. 90) make angles α', β', γ', with the three edges of the parallelopiped AB, AG, AD respectively; then

$$\cos\alpha' = \frac{a + h\cos\alpha + k\cos\beta}{AP}; \quad \cos\beta' = \frac{a\cos\alpha + h + k\cos\gamma}{AP};$$

$$\cos\gamma' = \frac{a\cos\beta + h\cos\gamma + k}{AP};$$

hence α', β', γ' are known.

Let a spherical surface (fig. 91) be described with centre A, cutting the three edges and AP, AO in the points a, b, c; p, O respectively;

$$\therefore\ ab = \alpha,\ ac = \beta,\ bc = \gamma;\ pa = \alpha',\ pb = \beta',\ pc = \gamma';$$

$$Oa = \theta,\ Ob = \phi,\ Oc = \psi;$$

$$\therefore \text{ in } \triangle Oac,\ \cos Oca = \frac{\cos\theta - \cos\beta\cos\psi}{\sin\beta\sin\psi};$$

$$\text{and } \cos acp = \frac{\cos\alpha' - \cos\beta\cos\gamma'}{\sin\beta\sin\gamma'}; \quad \therefore\ \angle Ocp \text{ is known};$$

$$\text{and } \cos Op = \cos\psi\cos\gamma' + \sin\psi\sin\gamma'\cos Ocp;$$

which determines the cosine of the inclination of Ap to the vertical, or the sine of its inclination to the horizon.

17. Let r_1, r_2, r_3 be the radii of the three circles which touch the sides terminated at A, B, C respectively (fig. 92); O_1, O_2, O_3 their centres; then AO_1, BO_2, CO_3 bisect the angles A, B, C; and if O_2a_1, O_3a_2 be drawn perpendicular to BC, we have

$$O_2O_3 = r_2 + r_3;\ a_1a_2 = \sqrt{(r_2+r_3)^3 - (r_2-r_3)^2} = 2\sqrt{r_2r_3};$$

$$\therefore\ BC = a = r_2\cot\frac{B}{2} + r_3\cot\frac{C}{2} + 2\sqrt{r_2r_3};$$

$$\text{or } r_2\cot\frac{B}{2} + r_3\cot\frac{C}{2} + 2\sqrt{r_2r_3} = r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right).$$

Similarly, $r_1 \cot\frac{A}{2} + r_3 \cot\frac{C}{2} + 2\sqrt{r_1 r_3} = r\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right)$,

$$r_1 \cot\frac{A}{2} + r_2 \cot\frac{B}{2} + 2\sqrt{r_1 r_2} = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right);$$

hence $r_1\left(\dfrac{\cot\frac{A}{2}}{\cot\frac{A}{2}+\cot\frac{C}{2}}\right) + \dfrac{2\sqrt{r_1 r_3}}{\cot\frac{A}{2}+\cot\frac{C}{2}} + \dfrac{r_3 \, . \cot\frac{C}{2}}{\cot\frac{A}{2}+\cot\frac{C}{2}}$

$$= r_2\frac{\cot\frac{B}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}} + \frac{2\sqrt{r_2 r_3}}{\cot\frac{B}{2}+\cot\frac{C}{2}} + \frac{r_3 \cot\frac{C}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}};$$

$$\therefore\ r_1\frac{\cos\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} + 2\sqrt{r_1 r_3}\,\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} + r_3\frac{\cos\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}}$$

$$= r_2\frac{\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} + 2\sqrt{r_2 r_3}\,\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} + r_3\frac{\cos\frac{C}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}};$$

$$\therefore\ r_1\cos^2\frac{A}{2} + 2\sqrt{r_1 r_3}\sin\frac{A}{2}\cos\frac{A}{2} + r_3\cot\frac{C}{2}\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= r_2\cos^2\frac{B}{2} + 2\sqrt{r_2 r_3}\sin\frac{B}{2}\cos\frac{B}{2} + r_3\cot\frac{C}{2}\sin\frac{B}{2}\cos\frac{B}{2};$$

hence $r_1\cos^2\frac{A}{2} + 2\sqrt{r_1 r_3}\sin\frac{A}{2}\cos\frac{A}{2} + r_3\sin^2\frac{A}{2}$

$$= r_2\cos^2\frac{B}{2} + 2\sqrt{r_2 r_3}\sin\frac{B}{2}\cos\frac{B}{2} + r_3\sin^2\frac{B}{2}.$$

or $\sqrt{r_1}\cos\frac{A}{2} + \sqrt{r_3}\sin\frac{A}{2} = \sqrt{r_2}\cos\frac{B}{2} + \sqrt{r_3}\sin\frac{B}{2}$.

Similarly, $\sqrt{r_3}\cos\frac{C}{2} + \sqrt{r_1}\sin\frac{C}{2} = \sqrt{r_2}\cos\frac{B}{2} + \sqrt{r_1}\sin\frac{B}{2}$;

$$\therefore \sqrt{r_1}\cos\frac{A}{2} + \sqrt{r_3}\left(\sin\frac{A}{2} - \sin\frac{B}{2}\right) = \sqrt{r_3}\cos\frac{C}{2} + \sqrt{r_1}\left(\sin\frac{C}{2} - \sin\frac{B}{2}\right),$$

or $\sqrt{r_1}\left(\cos\frac{A}{2} + \sin\frac{B}{2} - \sin\frac{C}{2}\right) = \sqrt{r_3}\left(\cos\frac{C}{2} + \sin\frac{B}{2} - \sin\frac{A}{2}\right)$.

Now $\cos\frac{A}{2} + \sin\frac{B}{2} - \sin\frac{C}{2} = 2\sin\frac{B}{4}\left(\sin\frac{2A+B}{4} + \sin\frac{2\pi - B}{4}\right)$

$$= 4\sin\frac{B}{4} . \sin\frac{\pi + A}{4} . \cos\frac{C}{4};$$

$$\therefore \; 4\sin\frac{B}{4}\sin\frac{\pi + A}{4}\cos\frac{C}{4}\sqrt{r_1} = 4\sin\frac{B}{4} . \sin\frac{\pi + C}{4} . \cos\frac{A}{4}\sqrt{r_3};$$

$$\therefore \; \frac{r_1}{r_3} = \left(\frac{1 + \tan\frac{C}{4}}{1 + \tan\frac{A}{4}}\right)^2,$$

$$\text{hence } r_1\left\{\cot\frac{A}{2} + \left(\frac{1 + \tan\frac{A}{4}}{1 + \tan\frac{C}{4}}\right)^2 . \cot\frac{C}{2} + 2\,\frac{\left(1 + \tan\frac{A}{4}\right)}{1 + \tan\frac{C}{4}}\right\}$$

$$= r\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right),$$

from which by reduction

$$r_1 = \frac{r}{2}\,\frac{\left(1 + \tan\frac{B}{4}\right)\left(1 + \tan\frac{C}{4}\right)}{1 + \tan\frac{A}{4}},$$

and similarly the values of r_2, r_3 are determined.

Next let AB, AC be produced to D, E, and R_1 the radius of the escribed circle touching BC; also let three circles be

described touching one another, and two of the lines BC, BD, CE; then we must suppose the triangle to have the angles $\pi - B$, $\pi - C$, and $-A$; and if ρ_1 be the radius of the circle which touches BD, CE,

$$\rho_1 = \frac{R_1}{2} \frac{\left(1 + \tan\frac{\pi - B}{4}\right)\left(1 + \tan\frac{\pi - C}{4}\right)}{1 - \tan\frac{A}{4}}$$

$$= \frac{2R_1}{\left(1 + \tan\frac{B}{4}\right)\left(1 + \tan\frac{C}{4}\right)\left(1 - \tan\frac{A}{4}\right)}.$$

ST JOHN'S COLLEGE. Dec. 1840. (No. XI.)

1. If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

2. Define duplicate ratio, and prove that similar triangles have to one another the duplicate ratio of their homologous sides.

3. Draw a straight line perpendicular to a plane from a given point above it.

4. Prove analytically that the angle in a semicircle is a right angle.

5. If ABC be a triangle where the angle C is a right angle, and if the sides CA, CB be produced, the straight lines bisecting the exterior angles at A and B when produced to meet, include an angle which is half a right angle.

6. If triangles be drawn with two sides coincident in direction, the locus of the centres of the inscribed circles is a straight line; and if the third side be also given in length, the locus of the centres of the circumscribed circles is a circle.

7. Through any fixed point A in the circumference of a circle draw chords AP, AQ at right angles to one another, and join PQ. If O be a point in PQ such that $PO = n \times PQ$, the locus of O is a circle.

8. In two given straight lines drawn from a point O, take points P, Q in one, and P', Q' in the other, so that OP, OQ, OP', OQ' are in harmonic progression; find the locus of the intersection of PQ', $P'Q$.

9. Considering the tangent as the limiting form of a secant, shew that the equation to the tangent to a parabola is

$$\rho = a \sec \frac{\theta'}{2} . \sec \left(\theta - \frac{\theta'}{2}\right),$$

the focus being pole, and θ' being the spiral angle of the point of contact.

10. Determine the position and dimensions of the Conic Section

$$3y^2 - 8ay + x^2 - 2\sqrt{3}xy + 8ax\sqrt{3} = 0,$$

and trace the curve

$$y^4 + x^2y^2 = a^2(x^2 - y^2).$$

11. Two cones whose vertical angles are supplementary, are placed with their vertices coincident, and their axes perpendicular; when a plane cuts them, compare the minor axis of the elliptic section of one, with the conjugate axis of the hyperbolic section of the other.

12. From A, B extremities of the diameter of a circle, draw chords AP, BP. Find the locus of the intersection of circles described on AP, BP as diameters.

13. A pyramid is constructed on a square base, having all its edges equal in length; find the inclination of two of the triangular faces to one another.

14. Describe two concentric and similarly situated ellipses not intersecting. Draw a tangent to the interior one, and at its intersections with the exterior ellipse, draw tangents to the latter. Find the locus of the intersections of the latter pairs of tangents.

15. Find the locus of the middle points of a system of parallel chords drawn between an hyperbola and the conjugate hyperbola.

16. If a pair of conjugate diameters of an ellipse when produced be asymptotes to an hyperbola, the points of the hyperbola at which a tangent to the hyperbola will also be a tangent to the ellipse lie in an ellipse similar to the given one.

SOLUTIONS TO (No. XI.)

1. Euclid, Prop. 24. Book I.

2. Euclid, Def. 10. Book V. and Prop. 19. Book VI.

3. Euclid, Prop. 11. Book. XI.

4. Let A (fig. 93) be the origin, and the diameter AB the axis of x; then the equation to the circle is $y^2 = 2ax - x^2$; and if $y = mx$ be the equation AP, at the point P

$$m^2x^2 = 2ax - x^2; \quad \therefore\ x = \frac{2a}{1+m^2}, \text{ and } y = \frac{2am}{1+m^2};$$

hence the equation to BP which passes through the points $2a$, 0; and $\dfrac{2a}{1+m^2}$, $\dfrac{2am}{1+m^2}$ becomes,

$$y = \frac{\dfrac{2am}{1+m^2}}{\dfrac{2a}{1+m^2} - 2a}(x - 2a),$$

or $y = -\dfrac{1}{m}(x - 2a)$ which is perpendicular to AP;

therefore $\angle APB$ is a right angle.

5. Let the two straight lines meet in D (fig. 94); then

$$\angle DAB = \tfrac{1}{2}(\pi - A); \qquad \angle DBA = \tfrac{1}{2}(\pi - B);$$

$$\therefore\ \angle ADB = \pi - \left\{\tfrac{1}{2}(\pi - A) + \tfrac{1}{2}(\pi - B)\right\} = \frac{A+B}{2} = \frac{\pi}{4}.$$

6. Let AB, AC (fig. 95) be two sides of the triangle including a given angle A; bisect $\angle BAC$ by the straight line AO, then the centre of the inscribed circle will be in the line AO, whatever be the lengths of AB, AC.

Next let O' be the centre of the circumscribing circle; then radius $AO' = \tfrac{1}{2}\dfrac{BC}{\sin A}$; and if BC and $\angle A$ be constant, AO' is constant; and the locus of O' is a circle whose centre is A, and radius $= \dfrac{BC}{2\sin A}$.

7. PQ (fig. 96) passes through the centre; $\therefore PQ = 2a$, $PO = 2na$; and $CO = (1-2n)\,a$ which is constant; hence the locus of O is a circle whose centre is C, and radius $= (1-2n)\,a$.

8. Let OPQ, $OP'Q'$ (fig. 97) be taken for the axes of x and y respectively;

$$\text{let } OP = a,\ \ OQ = a',\ \ OP' = b,\ \ OQ' = b';$$

then the equation to PQ' is

$$\frac{x}{a} + \frac{y}{b'} = 1;$$

and the equation to $P'Q$ is

$$\frac{x}{a'} + \frac{y}{b} = 1;$$

hence at their point of intersection

$$\left(\frac{1}{a} - \frac{1}{a'}\right) x - \left(\frac{1}{b} - \frac{1}{b'}\right) y = 0;$$

$$\text{but } \frac{1}{a} - \frac{1}{a'} = \frac{1}{b} - \frac{1}{b'}; \quad \therefore\ x - y = 0;$$

hence the intersection will always be found in the straight line whose equation is $x - y = 0$, which bisects the angle POQ'.

9. Let $r = p \sec(\theta - \beta)$ be the polar equation to the secant; hence $p = r\cos(\theta - \beta)$; and if θ', θ'' be the polar angles of the points of intersection of the secant and curve, r', r'' the corresponding focal distances, we have

$$p = r'\cos(\theta' - \beta), \qquad p = r''\cos(\theta'' - \beta);$$

$$r' = \frac{2a}{1 + \cos\theta'}; \quad r'' = \frac{2a}{1 + \cos\theta''};$$

$$\therefore\ \frac{\cos(\theta' - \beta)}{1 + \cos\theta'} = \frac{\cos(\theta'' - \beta)}{1 + \cos\theta''};$$

$$\text{or } (1 + \cos\theta'')(\cos\theta'\cos\beta + \sin\theta'\sin\beta)$$

$$= (1 + \cos\theta')(\cos\theta''\cos\beta + \sin\theta''\sin\beta);$$

$$\text{hence } (\cos\theta' - \cos\theta'')\cos\beta = \{\sin\theta'' - \sin\theta' + \sin(\theta'' - \theta')\}\sin\beta;$$

$$\therefore \sin\frac{\theta' + \theta''}{2} = \left(\cos\frac{\theta'' + \theta'}{2} + \cos\frac{\theta'' - \theta'}{2}\right)\tan\beta$$
$$= 2\cos\frac{\theta'}{2}\cos\frac{\theta''}{2}\tan\beta;$$
$$\text{hence } \tan\beta = \tfrac{1}{2}\left(\tan\frac{\theta'}{2} + \tan\frac{\theta''}{2}\right);$$

and when θ', θ'' approach to one another,

$$\tan\beta = \tan\frac{\theta'}{2}, \text{ or } \beta = \frac{\theta'}{2};$$

also $p = r'\cos(\theta' - \beta) = r'\cos\frac{\theta'}{2} = a\sec^2\frac{\theta'}{2}\,.\cos\frac{\theta'}{2} = a\sec\frac{\theta'}{2}$;

hence the polar equation to the tangent becomes

$$r = a\sec\frac{\theta'}{2}\sec\left(\theta - \frac{\theta'}{2}\right).$$

10. The equation is that of a parabola.

Transform the origin to a point α, β in the curve by making $x = x' + \alpha$, $y = y' + \beta$;

$$\therefore 3(y' + \beta)^2 - 2\sqrt{3}(x' + \alpha)(y' + \beta) + (x' + \alpha)^2$$
$$- 8a(y' + \beta) + 8a\sqrt{3}(x' + \alpha) = 0;$$
$$\text{hence } 3y'^2 - 2\sqrt{3}x'y' + x'^2 + d'y' + e'x' = 0,$$
$$\text{where } \phi(\alpha, \beta) = 0; \quad 6\beta - 2\sqrt{3}\alpha - 8a = d',$$
$$2\alpha - 2\sqrt{3}\beta + 8a\sqrt{3} = e'.$$

Again, transform the equation to polar co-ordinates by putting $x' = \rho\cos\theta$, $y' = \rho\sin\theta$;

$$\therefore (\sqrt{3}\sin\theta - \cos\theta)^2\rho + (d'\sin\theta + e'\cos\theta) = 0;$$
$$\text{and if } \frac{d'}{e'} = \frac{1}{\sqrt{3}} = \tan\delta = \tan\frac{\pi}{6};$$
$$4\sin^2(\theta - \delta)\rho + e'\sec\delta\,.\cos(\theta - \delta) = 0;$$

which is the equation to a parabola whose axis makes an $\angle 30$ with the axis of x, and the latus rectum $= -\dfrac{e'\sec\delta}{4}$.

$$\text{Now } \frac{6\beta - 2\sqrt{3}a - 8a}{2a - 2\sqrt{3}\beta + 8a\sqrt{3}} = \frac{1}{\sqrt{3}};$$

$$\therefore \beta\sqrt{3} - \alpha = 2a\sqrt{3};$$

$$\text{and } e' = 8a\sqrt{3} - 2(2a\sqrt{3}) = 4a\sqrt{3};$$

$$\therefore \text{latus rectum} = -\frac{4a\sqrt{3}}{4} \cdot \frac{2}{\sqrt{3}} = -2a;$$

$$\phi(\alpha, \beta) = 3\beta^2 - 2\sqrt{3}\alpha\beta + \alpha^2 - 8a\beta + 8a\sqrt{3}.\alpha = 0;$$

$$\therefore 8a(\beta - \sqrt{3}\alpha) = (\sqrt{3}\beta - \alpha)^2 = 12a^2,$$

$$\text{or } 2(\beta - \sqrt{3}\alpha) = 3a;$$

$$\text{and } 6\beta - 2\sqrt{3}\alpha = 12a,$$

$$\therefore 4\beta = 9a; \text{ and } \beta = \frac{9a}{4};$$

$$\text{also } 2\sqrt{3}\alpha = 2\beta - 3a = \frac{9a}{2} - 3a = \frac{3a}{2};$$

$$\text{hence } \alpha = \frac{\sqrt{3}\,a}{4};$$

α, β are the co-ordinates of the vertex of the parabola, and the negative sign of the latus rectum shews that it extends indefinitely on the negative side of the origin.

(2) To trace the curve $y^4 + x^2y^2 = a^2(x^2 - y^2)$, let the equation be transformed to polar co-ordinates by putting

$$x = \rho\cos\theta, \quad y = \rho\sin\theta; \quad \therefore \rho^2 = a^2(\cot^2\theta - 1);$$

and when $\theta = 0$, $\rho = \infty$: also

$$\rho^2\sin^2\theta = a^2\cos 2\theta = a^2:$$

hence the value of the ordinate $= \pm a$ when x is infinite, or the curve has two asymptotes parallel to the axis of x at a distance a from it.

As θ increases from 0 to $\frac{\pi}{4}$; ρ diminishes from infinity to 0, and the curve passes through the origin.

Since by putting $-x$ for x, or $-y$ for y, the equation to the curve is unaltered, it is symmetrical both with respect to the axis of x and y, and has a point of contrary flexure at the origin A. The curve is of the form traced in fig. 98.

11. Let AP (fig. 99) $= p$ be the perpendicular upon the cutting plane PBC; θ the inclination of AP to the axis of the cone; draw BD, CE perpendicular to the axis; and let $2a$ be the vertical angle of the cone, $2b$ the minor axis of the section, then

$$4b^2 = BD \,.\, CE = 2p \sec(\theta - a) \sin a \left\{2p \sec(\theta + a) \sin a\right\}$$

$$= \frac{4p^2 \sin^2 a}{\cos(\theta + a)\cos(\theta - a)} = \frac{4p^2 \sin^2 a}{\cos^2\theta - \sin^2 a}.$$

Similarly, if θ' be the angle which AP makes with the axis of the second cone whose vertical angle is $2a'$;

$$4b'^2 = \frac{4p^2 \sin^2 a'}{\sin^2 a' - \cos^2\theta'}:$$

and when $2a' = \pi - 2a$, and $\theta' = \frac{\pi}{2} + \theta$;

$$\frac{4b'^2}{4b^2} = \cot^2 a;$$

which is the ratio required.

12. Draw PM (fig. 83) perpendicular to AB; then since the angles AMP, BMP are right angles, the semicircles described on AP and BP both pass through M which is their point of intersection; and the locus of the intersection of the circles is the line AB. The same is equally true when P is any point whatever, and the locus is independent of the circle APB.

13. Let A (fig. 100) be one of the angles of the base; with centre A describe a spherical surface intersecting the two triangular faces and the base in ab, ac, bc; then

$$ab = ac = 60^0, \quad bc = 90^0;$$

$$\therefore \cos \angle bac = \frac{\cos 90 - \cos 60 \,.\, \cos 60}{\sin 60 \,.\, \sin 60} = -\tfrac{1}{3};$$

and $\angle bac$ which measures the inclination between two triangular faces $= \cos^{-1}(-\tfrac{1}{3}) = \pi - \cos^{-1}\tfrac{1}{3}$.

14. Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and $\dfrac{x^2}{a'^2} + \dfrac{y^2}{b'^2} = 1$ be the equations to the two ellipses; then the equation to a tangent at a point x', y' of the first ellipse is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1. \quad (1)$$

Let a pair of tangents be drawn from a point X, Y to the second ellipse, then the equation to the line joining the points of contact is $\dfrac{Xx}{a'^2} + \dfrac{Yy}{b'^2} = 1$; and in order that this may coincide with equation (1) we must have

$$\frac{x'}{a^2} = \frac{X}{a'^2}, \quad \frac{y'}{b^2} = \frac{Y}{b'^2}, \quad \text{and} \quad \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1;$$

$$\therefore \left(\frac{aX}{a'^2}\right)^2 + \left(\frac{bY}{b'^2}\right)^2 = 1,$$

which is the equation to an ellipse whose semiaxes are

$$\frac{a'^2}{a}, \quad \frac{b'^2}{b}.$$

15. Transform the origin to a point α, β; then the equation to the hyperbola is

$$\left(\frac{x+\alpha}{a}\right)^2 - \left(\frac{y+\beta}{b}\right)^2 = 1;$$

and if $y = mx$ be the equation to a chord passing through the origin, when the chord meets the hyperbola

$$\frac{(x+\alpha)^2}{a^2} - \left(\frac{mx+\beta}{b}\right)^2 = 1\,;$$

and the values of x in which the chord meets the conjugate hyperbola will be determined from the equation

$$\frac{(x'+\alpha)^2}{a^2} - \left(\frac{mx'+\beta}{b}\right)^2 = -1\,;$$

but when the origin is the middle point of the chord $x' = -x$;

$$\therefore\ \left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)x^2 + \left(\frac{2\alpha}{a^2} - \frac{2m\beta}{b^2}\right)x + \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 1 = 0\,;$$

$$\text{and}\ \left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)x^2 - 2\left(\frac{\alpha}{a^2} - \frac{m\beta}{b^2}\right)x + \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} + 1 = 0\,;$$

$$\therefore\ 4\left(\frac{\alpha}{a^2} - \frac{m\beta}{b^2}\right)x - 2 = 0:$$

and by addition

$$\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)x^2 + \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} = 0\,;$$

hence by eliminating x,

$$\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) + 4\left(\frac{\alpha}{a^2} - \frac{m\beta}{b^2}\right)^2\left(\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2}\right) = 0,$$

which gives a relation between α and β, and is the equation required.

16. Let the equation to the ellipse be $\dfrac{x'^2}{a'^2} + \dfrac{y'^2}{b'^2} = 1$; and let x, y be the co-ordinates of a point in the hyperbola at which a tangent to the hyperbola will be also a tangent to the ellipse, then

$$\frac{Xx'}{a'^2} + \frac{Yy'}{b'^2} = 1\,; \ \text{and}\ \frac{a'^2}{x'} = 2x,\ \frac{b'^2}{y'} = 2y\,;$$

$$\therefore \left(\frac{a'}{2x}\right)^2 + \left(\frac{b'}{2y}\right)^2 = 1\,;$$

$$\text{or } a'^2y^2 + b'^2x^2 = 4x^2y^2, \text{ and } xy = m^2\,;$$

$$\therefore a'^2y^2 + b'^2x^2 = 4m^4,$$

which shews that the point x, y lies in an ellipse whose semi-conjugate diameters are $\frac{2m^2}{b'}$ and $\frac{2m^2}{a'}$, or the ellipse is similar to the original one.

In this example an ellipse has been found similar to the original one which will pass through the points denoted by x, y; but the two equations $a'^2y^2 + b'^2x^2 = 4m^4$, and $xy = m^2$ may be combined in every possible way, giving different curves which will intersect the hyperbola in the required points. The curve may be reduced to a right line for

$$a'^2y^2 \pm 2a'b'xy + b'^2x^2 = 4m^4 \pm 2a'b'm^2,$$

$$\text{or } a'y \pm b'x = \pm m\sqrt{4m^2 \pm 2a'b'}\,;$$

either of which pair of straight lines will intersect the hyperbola in the four points required.

ST JOHN'S COLLEGE. Dec. 1841. (No. XII.)

1. The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to the sum of the rectangles contained by its opposite sides.

2. Four circles are drawn, of which each touches one side of a quadrilateral figure and the adjacent sides produced; shew that the centres of these four circles will all lie in the circumference of a circle.

3. If $ABCD$ be any quadrilateral, M, N, P, Q the bisections of its sides, prove that

$$AC^2 + BD^2 = 2(MP^2 + NQ^2).$$

4. Three circles, whose radii form a geometric progression, having 2 as a common ratio, touch each other; find the angles of the enveloping triangle.

5. The lines joining the angles of a triangle with the points in which the escribed circles touch the opposite sides, meet in a point. Shew also that if the base and the sum of the other sides is constant, the locus of the centre of the escribed circle touching the base is an ellipse.

6. Two semicircles are described on the segments of the diameter of a semicircle whose radius is r; shew that the locus of the centre of the circle touching these three semicircles is an ellipse whose semiaxes are $\frac{4r}{3}$, and $\frac{2r}{\sqrt{3}}$.

7. Tangents are drawn to an ellipse so that the product of the trigonometrical tangents of their inclinations to the major axis is constant; prove that the locus of their intersection is a conic section.

8. Through A the common vertex of two similar ellipses ABB', ADD' whose greater axes coincide, draw chords ABD, $AB'D'$ and join BB', DD'; BB' will be parallel to DD'.

9. If A be the elliptic area contained by two semi-

diameters including an angle α, and B that contained between two semi-diameters at right angles to the first, then

$$\left(\frac{a}{b}+\frac{b}{a}\right)\cot\alpha=\cot\frac{2A}{ab}+\cot\frac{2B}{ab}.$$

10. The straight lines drawn from any point in an equilateral hyperbola to the extremities of any diameter, are inclined at equal angles to the asymptotes.

11. From any fixed point P in an hyperbola, draw PA, PB parallel to its asymptotes, and from another fixed point Q draw any straight line cutting these in a, b and the curve in p; then $pa \propto pb$.

12. A, B are two fixed points, P, Q any two points in the line AB or AB produced, such that

$$k.\frac{PA}{m}.\frac{QB}{n}+l.\frac{QB}{n}-g\frac{PA}{m}-h=0;$$

find two other fixed points M, N, such that

$$\frac{n}{QN}-\frac{m}{PM}=K;$$

and find the values of K, AM, AN.

13. If any number of quadrilaterals inscribed in a circle, have a common side, and the sides adjacent to this be produced to meet; the lines joining the point of concourse, with the intersection of the diagonals of the quadrilateral shall all meet in the same point.

SOLUTIONS TO (No. XII.)

1. Euclid, Prop. D. Book vi.

2. Let $ABCD$ (fig. 101) be the quadrilateral figure; O the centre of the circle touching AB and the two sides CB, DA produced; then OA, OB bisect the exterior angles of the figure, and $\angle AOB = \pi - (OAB + OBA)$

$$= \pi - \left\{\tfrac{1}{2}(\pi - A) + \tfrac{1}{2}(\pi - B)\right\} = \frac{A+B}{2}.$$

Similarly if O' be the centre of the circle touching CD and the two sides BC, AD produced,

$$\angle DO'C = \frac{C+D}{2} = \frac{2\pi - (A+B)}{2} = \pi - \frac{A+B}{2} = \pi - AOB;$$

$$\text{hence } \angle AOB + \angle DO'C = \pi.$$

Also if P, P' be the centres of the circles touching the sides AD, BC respectively, $\angle APD + \angle BP'C = \pi$; and the opposite angles of the quadrilateral figure $OPO'P'$ are together equal to two right angles; hence a circle may be described about $OPO'P'$; and the centres lie in the circumference of this circle.

3. MN and PQ (fig. 102) are each parallel to AC, and equal to $\dfrac{AC}{2}$; also QM and PN are each parallel to BD, and equal to $\dfrac{BD}{2}$; hence $MNPQ$ is a parallelogram whose diagonals are MP, NQ;

$$\therefore\ MP^2 + NQ^2 = 2\,MN^2 + 2\,QM^2 = \tfrac{1}{2}AC^2 + \tfrac{1}{2}BD^2;$$

$$\therefore\ AC^2 + BD^2 = 2(MP^2 + NQ^2).$$

4. Let A, B, C (fig. 103) be the centres of the three circles whose radii are as 4, 2, 1 respectively; ab the direction of the common tangent to the circles whose centres are A, B; ac the direction of the common tangent to the circles whose

centres are A, C; bc the common tangent to the circles whose centres are B, C; join AB, AC, BC and produce them to meet ab, ac, bc respectively in d, e, f; then the triangle abc will touch the three circles with the sides bc, and the two sides ab, ac produced; but there cannot be any triangle which envelopes the circles and touches them with its three sides.

Let r_1, r_2, r_3 be the radii of the three circles whose centres are A, B, C; $\angle Cfc = \theta$, $\angle Aec = \phi$; then

$$\sin\theta = \frac{r_2 - r_3}{r_3 + r_2} = \tfrac{1}{3};$$

$$\sin Adb = \frac{r_1 - r_2}{r_1 + r_2} = \frac{4 - 2}{4 + 2} = \tfrac{1}{3} = \sin\theta;$$

$$\angle ABC - \angle Bfc = \pi - \angle abc - \angle Adb;$$

$$\text{or } \angle abc = \pi - ABC = \pi - B.$$

$$\text{Again } \angle ead + \angle eAd = \angle Adb + \angle Aec = \theta + \phi,$$

$$\text{or } \angle cab = \theta + \phi - \angle CAB;$$

$$\therefore \angle acb = \pi - \{\pi - B + \theta + \phi - A\} = A + B - (\theta + \phi);$$

$$\text{but } \sin\theta = \tfrac{1}{3};\ \sin\phi = \frac{r_1 - r_3}{r_1 + r_3} = \frac{3}{5};\ BC = 3r_1,\ AC = 5r_1,\ AB = 6r_1;$$

$$\text{or } \sin A = \frac{2\sqrt{14}}{15};\ \sin B = \frac{2\sqrt{14}}{9};\ \sin C = \frac{4\sqrt{14}}{15};$$

$$\therefore \angle b = \pi - B = \pi - \sin^{-1}\left(\frac{2\sqrt{14}}{9}\right);$$

$$\angle a = \sin^{-1}\tfrac{1}{3} + \sin^{-1}\tfrac{3}{5} - \sin^{-1}\frac{2\sqrt{14}}{15};$$

$$\angle c = \sin^{-1}\frac{2\sqrt{14}}{15} + \sin^{-1}\frac{2\sqrt{14}}{9} - \sin^{-1}\tfrac{1}{3} - \sin^{-1}\tfrac{3}{5}.$$

5. (1) Let O (fig. 104) be the centre of the circle which touches the side BC and the other two sides produced;

M, N, P the three points of contact of the escribed circles with the sides; then

$$BM = S - c; \quad BP = S - a; \quad CN = S - a;$$

and if BC, BA be the co-ordinate axes, the equations to AM, CP are

$$\frac{x}{S-c} + \frac{y}{c} = 1; \quad \frac{x}{a} + \frac{y}{S-a} = 1;$$

and the co-ordinates of N are

$$\frac{a}{b}(AN), \text{ and } \frac{c}{b}(CN),$$

$$\text{or } \frac{a}{b}(S-c), \text{ and } \frac{c}{b}(S-a);$$

hence the equation to BN is

$$\frac{y}{x} = \frac{c(S-a)}{a(S-c)};$$

and for the points of intersection of AM, BN we have

$$x = \frac{a(S-c)}{S}, \quad y = \frac{c(S-a)}{S};$$

which are the same as the co-ordinates of the point of intersection of BN, CP; therefore AM, BN, CP meet in the same point.

(2) Next let BC be the axis of x; $BM = x$, $MO = y$; $BC = a$; $BA + AC = s$; then

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin(B+C)} = \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}} = \frac{1 + \tan\frac{B}{2}\tan\frac{C}{2}}{1 - \tan\frac{B}{2}\tan\frac{C}{2}} = \frac{s}{a};$$

$$\therefore \tan\frac{B}{2}\tan\frac{C}{2} = \frac{s-a}{s+a}; \quad \text{and } \frac{y}{x} = \cot\frac{B}{2};$$

$$\frac{y}{a-x} = \cot\frac{C}{2}; \quad \text{hence } \frac{ax - x^2}{y^2} = \left(\frac{s-a}{s+a}\right);$$

$$\text{and } y^2 = \frac{s+a}{s-a}(ax - x^2)$$

which is the equation to an ellipse having the base for its axis minor, and axis major $= \sqrt{\dfrac{s+a}{s-a}} \,.\, a.$

6. Let AB (fig. 105) be the diameter of the given semi-circle whose centre is O, and radius r; C a point in AB the centre of another semi-circle whose radius is r'; D the centre and ρ' the radius of any other circle which touches the two former semi-circles; P the centre and ρ the radius of the circle which touches the three circles whose centres are C, O, D; draw PM perpendicular to AB, and let $AM = x$, $MP = y$; x', y' the co-ordinates of D; then (Solutions of Trigonometrical Problems. Ex. 18. No. 15)

$$\frac{x}{\rho} = \frac{r + r'}{r - r'}; \quad \frac{y}{\rho} - \frac{y'}{\rho'} = 2;$$

and when the point D is in AB, $y' = 0$;

$$\therefore \frac{x}{\rho} = \frac{r + r'}{r - r'}; \quad \frac{y}{\rho} = 2;$$

and $(x - r')^2 + y^2 = (\rho + r')^2$, $\quad \therefore x^2 - 2r'x + y^2 = \rho^2 + 2\rho r'$;

and $\dfrac{r'}{r} = \dfrac{x - \rho}{x + \rho}$; $\quad \therefore x^2 + y^2 - \rho^2 = 2r'(x + \rho) = 2r(x - \rho)$;

$$\text{or } x^2 + y^2 - \frac{y^2}{4} = 2r\left(x - \frac{y}{2}\right);$$

$$\text{hence } x^2 - 2rx + \frac{3y^2}{4} + ry = 0;$$

$$\text{or } (x - r)^2 + \tfrac{3}{4}\left(y + \frac{2r}{3}\right)^2 = \frac{4r^2}{3};$$

which is the equation to an ellipse whose semi-axes are $\dfrac{4r}{3}$, and $\dfrac{2r}{\sqrt{3}}$.

If P be the centre of the n^{th} circle inscribed between the circles whose centres are C, O, D; $x_n y_n$ its co-ordinates; we have

$$\frac{x_n}{\rho_n} = \frac{r + r'}{r - r'}; \quad \frac{y_n}{\rho_n} - \frac{y'}{\rho'} = 2n; \text{ or } \frac{y_n}{\rho_n} = 2n, \text{ since } y' = 0;$$

$$\text{and } (x_n - r')^2 + y_n^2 = (\rho_n + r')^2;$$

$$\therefore\ x_n^2 - 2r' x_n + y_n^2 = \rho_n^2 + 2r' \rho_n;$$

$$\text{and } x_n^2 + y_n^2 - \rho_n^2 = 2r' (x_n + \rho_n) = 2r (x_n - \rho_n);$$

$$\therefore\ x_n^2 + y_n^2 - \frac{(y_n)^2}{4n^2} = 2r \left(x_n - \frac{y_n}{2n}\right);$$

$$\therefore\ x_n^2 + \frac{(4n^2 - 1)}{4n^2} y^2 - 2r x_n + \frac{2r}{2n} y_n = 0;$$

$$\therefore\ (x_n - r)^2 + \left(\frac{4n^2 - 1}{4n^2}\right) \left(y - \frac{2n}{4n^2 - 1} r\right)^2 = \frac{4n^2}{4n^2 - 1} r^2;$$

which is the equation to an ellipse whose semi-axes are

$$\frac{4n^2}{4n^2 - 1} r, \quad \sqrt{\frac{4n^2}{4n^2 - 1}}\, r;$$

and co-ordinates of the centre r, and $\dfrac{2n}{4n^2 - 1} r$.

7. The equation to a tangent at the point x', y' of an ellipse is $\dfrac{x'x}{a^2} + \dfrac{y'y}{b^2} = 1$; and if tangents be drawn from a point X, Y without the ellipse,

$$\frac{x'X}{a^2} + \frac{y'Y}{b^2} = 1.$$

Let m be the tangent of the angle which one of the tangents makes with the axis of x; then

$$m = -\frac{b^2 x'}{a^2 y'}, \text{ and } Y - mX = \frac{b^2}{y'};$$

$$\text{but } \frac{x'}{a} = -\frac{am}{b} \cdot \frac{y'}{b}; \quad \therefore \left\{\left(\frac{am}{b}\right)^2 + 1\right\} \frac{y'^2}{b^2} = 1,$$

$$\text{or} \quad \frac{b^2}{y'} = \sqrt{b^2 + a^2 m^2}\,; \quad \text{hence} \quad Y - mX = \sqrt{b^2 + a^2 m^2},$$

$$\text{or} \quad m^2 (X^2 - a^2) - 2XYm + Y^2 - b^2 = 0\,; \quad (1)$$

and the product of the two values of $m = \alpha$;

$$\therefore \frac{Y^2 - b^2}{X^2 - a^2} = \alpha, \quad \text{or} \quad Y^2 - \alpha X^2 = b^2 - a^2 \alpha\,;$$

which is the equation to a conic section whose centre is the centre of the given ellipse.

From equation (1) it appears that if the sum of the two values of m be constant $= \beta$; $\dfrac{2XY}{X^2 - a^2} = \beta$, which is the equation to a hyperbola.

8. Taking the polar equation from the vertex, if $\angle BAC$ (fig. 106) $= \theta$; $\angle B'AC = \theta'$, $AB = \rho$, $AD = \rho'$; $2a$, $2a'$ the major-axes, and e the eccentricity; then

$$\rho = \frac{2a(1 - e^2)\cos\theta}{1 - e^2 \cos^2\theta}, \quad \rho' = \frac{2a'(1 - e^2)\cos\theta}{1 - e^2\cos^2\theta}\,;$$

$$\therefore \frac{AB}{AD} = \frac{\rho}{\rho'} = \frac{a}{a'}. \quad \text{Similarly} \quad \frac{AB'}{AD'} = \frac{a}{a'}\,;$$

$$\therefore \frac{AB}{AD} = \frac{AB'}{AD'}, \quad \text{or} \quad \frac{AB}{AB'} = \frac{AD}{AD'}\,;$$

hence BB' is parallel to DD'.

9. If x, y, x', y' be the co-ordinates of P, P' in the ellipse (fig. 107); then

$$\text{area } PCP' = \text{area } ACP' - ACP$$

$$= \frac{ab}{2}\left(\cos^{-1}\frac{x'}{a} - \cos^{-1}\frac{x}{a}\right),$$

$$\text{or } A = \frac{ab}{2}\left(\tan^{-1}\frac{ay'}{bx'} - \tan^{-1}\frac{ay}{bx}\right);$$

$$\therefore \tan\frac{2A}{ab} = \frac{\frac{a}{b}\frac{y'}{x'} - \frac{a}{b}\frac{y}{x}}{1 + \left(\frac{a}{b}\right)^2 \cdot \frac{y}{x} \cdot \frac{y'}{x'}} = \frac{a}{b}\,\frac{(\tan\theta' - \tan\theta)}{1 + \left(\frac{a}{b}\right)^2 \tan\theta\tan\theta'},$$

where $\angle PCA = \theta$, $\angle P'CA = \theta'$, and $\theta' - \theta = \alpha$;

$$\therefore \frac{\tan\theta' - \tan\theta}{1 + \tan\theta\tan\theta'} = \tan\alpha,$$

$$\text{or } \cot\frac{2A}{ab} = \frac{b}{a}\cot\alpha\left\{\frac{1 + \left(\frac{a}{b}\right)^2\tan\theta\tan\theta'}{1 + \tan\theta\tan\theta'}\right\}.$$

$$\text{Similarly, } \cot\frac{2B}{ab} = \frac{b}{a}\cot\alpha\left\{\frac{1 + \left(\frac{a}{b}\right)^2\cot\theta\cot\theta'}{1 + \cot\theta\cot\theta'}\right\}$$

$$= \frac{b}{a}\cot\alpha\left\{\frac{\left(\frac{a}{b}\right)^2 + \tan\theta\tan\theta'}{1 + \tan\theta\tan\theta'}\right\};$$

$$\therefore \cot\frac{2A}{ab} + \cot\frac{2B}{ab} = \frac{b}{a}\cot\alpha\left(1 + \frac{a^2}{b^2}\right) = \left(\frac{a}{b} + \frac{b}{a}\right)\cot\alpha.$$

10. The straight lines drawn from any point of a conic section to the extremities of a diameter are parallel to two conjugate diameters; and in the equilateral hyperbola every pair of conjugate diameters is equally inclined to the asymptotes; hence the straight lines drawn to the extremities of a diameter are equally inclined to the asymptotes.

11. Let the asymptotes (fig. 108) be taken for the axes; α, β, α_1, β_1 the co-ordinates of P, Q respectively; α', β' the co-ordinates of p; α, β'', α'', β the co-ordinates of b, a respectively; then the equation to pba is

$$y - \beta_1 = m(x - \alpha_1), \text{ and } \beta' - \beta_1 = m(\alpha' - \alpha_1),$$

$$\beta'' - \beta_1 = m(\alpha - \alpha_1); \quad \therefore \beta' - \beta'' = m(\alpha' - \alpha);$$

$$\text{and } \frac{pb}{pa} = \frac{\beta' - \beta''}{\beta' - \beta} = \frac{m(\alpha' - \alpha)}{\beta' - \beta}.$$

Now $\alpha\beta = a^2$, $\alpha'\beta' = a^2$, from the equation to the hyperbola;

$$\therefore \beta' - \beta = \frac{a^2}{\alpha'} - \frac{a^2}{\alpha} = -\frac{a^2}{\alpha\alpha'}(\alpha' - \alpha);$$

$$\text{hence } \frac{pb}{pa} = -m\frac{\alpha\alpha'}{a^2};$$

$$\text{but } \beta' - \beta_1 = \frac{a^2}{\alpha'} - \frac{a^2}{\alpha_1} = m(\alpha' - \alpha_1);$$

$$\therefore m = -\frac{a^2}{\alpha'\alpha_1}; \quad \text{hence } \frac{pb}{pa} = \frac{\alpha}{\alpha_1},$$

which is independent of m, or $pb \propto pa$.

12. Let AM (fig. 109) $= m\alpha$, $BN = n\beta$, $PM = mx$, $QN = ny$;

$$\therefore k(x + \alpha)(y + \beta) + l(y + \beta) - g(x + \alpha) - h = 0,$$

or $kxy + (k\alpha + l)y + (k\beta - g)x + k\alpha\beta + l\beta - g\alpha - h = 0$;

and if α, β be assumed so that

$$k\alpha\beta + l\beta - g\alpha - h = 0, \text{ and } k\alpha + l = -(k\beta - g);$$

$$\therefore k\beta = g - l - k\alpha;$$

$$\text{hence } -k\alpha(k\alpha + l) + l(g - l - k\alpha) - hk = 0;$$

$$\therefore k^2\alpha^2 + 2kl\alpha + l^2 = gl - hk;$$

$$\text{or } k\alpha + l = \pm\sqrt{gl - hk},$$

$$\text{and } kxy + (k\alpha + l)(y - x) = 0;$$

$$\text{hence } \frac{1}{y} - \frac{1}{x} = \frac{k}{k\alpha + l} = \pm\frac{k}{\sqrt{gl - hk}} = K.$$

$$\text{Also} \quad AM = m\alpha = m\left(\frac{\sqrt{gl - hk} - l}{k}\right),$$

$$BN = n\beta = n\left(\frac{g - \sqrt{gl - hk}}{k}\right);$$

and $AN = AB - BN$ is known.

13. Let AB (fig. 110) be the common side of the quadrilateral $ABCD$; produce AD, BC to meet in E; and AB, DC to meet in G; then EF is the chord of contact of two tangents drawn from G, a point in AB (App. II. Art. 61); but if pairs of tangents be drawn from any point in the straight line AB, the chords of contact will all pass through a fixed point; hence EF pass through a fixed point.

ST JOHN'S COLLEGE. DEC. 1842. (No. XIII.)

1. WHAT are the objections to Euclid's theory of parallels? State any other theory that you recollect.

How did Legendre escape the difficulty by an analytical process.

2. Similar triangles are to one another in the duplicate ratio of their homologous sides.

3. If a straight line be at right angles to a plane, every plane which passes through it is at right angles to that plane.

4. If two circles intersect, the common chord produced bisects the common tangent.

5. If A be the vertex, S the focus, PSp a focal chord in a parabola, the triangle PAp varies as $\sqrt{Pp}$.

6. Let the three perpendiculars from the angles of a triangle ABC on the opposite sides meet in P; a circle described so as to pass through P and any two of the points A, B, C is equal to the circumscribing circle of the triangle.

7. APB is a segment of a circle. Take any point P, and let PC be drawn dividing the angle APB in a given ratio; shew that the dividing lines all pass through a point, and find its co-ordinates.

8. The locus of the foci of all ellipses described with their major-axes parallel to the base of an isosceles triangle, and touching its three sides is a circle.

9. Given a circle and two points A, B exterior to it, find a point X in the circle such that if XA, XB be drawn cutting the circle in P, Q; PQ shall be parallel to AB.

10. If r be the radius of the inscribed, r_1, r_2, r_3, r_4 the radii of the escribed spheres of a triangular pyramid,

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}.$$

11. If a tangent be drawn to the interior one of two similar concentric ellipses whose axes are in the same straight lines, meeting the exterior one in P, Q, and at P, Q tangents be drawn to this latter intersecting in R, the locus of R is a similar ellipse.

12. CP, CQ are conjugate diameters of an ellipse; on PQ describe an equilateral triangle PQR, and find the locus of R.

13. Find the area included by normals to an hyperbola, which pass through the foci of the conjugate hyperbola.

14. From P, the point of concourse of two tangents to a parabola, PQ, PQ', draw $PABC$ meeting the curve in A, C, and the line QQ' in B; then PC is divided harmonically.

15. Two equal ellipses which have the same centre, have their axes inclined at a given angle, find the angle between the curves where they intersect.

16. A plane is drawn through a tangent to the circular base of a right cone cutting off an oblique cone whose volume is $\frac{1}{n}$th of the original cone, if θ be the inclination of the cutting plane to the side of the cone, 2α the vertical angle, $\sin 2\alpha \cot\theta = n^{\frac{2}{3}} - \cos 2\alpha$.

17. If a hexagon be described about a circle or ellipse, the three lines joining opposite angular points intersect in a point.

18. Give the geometrical construction of the following equations:

$$(1) \quad y^2 - 4xy + 5x^2 - 2y + 5 = 0.$$

$$(2) \quad y^2 - 2xy + x^2 - 2y - 2x - 3 = 0.$$

19. If a conic section be touched by four straight lines, the locus of its centre is a straight line.

SOLUTIONS TO (No. XIII.)

1. See Potts' Euclid, p. 50.

2. Euclid, Prop. 19, Book vi.

3. Euclid, Prop. 18, Book xi.

4. Let PQ (fig. 111) be the common tangent, AB the common chord; produce AB to meet PQ in T; then

$$TP^2 = TB \cdot TA = TQ^2, \text{ or } TP = TQ;$$

hence PQ is bisected in T.

5. Let ASB (fig. 112) be the axis; $\angle PSB = \theta$; then

$$SP = \frac{2a}{1 - \cos\theta}; \quad Sp = \frac{2a}{1 + \cos\theta}; \quad \therefore\ Pp = \frac{4a}{\sin^2\theta};$$

$$\text{and } \Delta\, PAp = \tfrac{1}{2} AS \cdot Pp \sin\theta = \frac{2a^2}{\sin\theta} = a^2 \sqrt{\frac{Pp}{a}} \propto \sqrt{Pp}.$$

6. $\angle APB$ (fig. 113) $= \pi - (\angle PAB + \angle PBA)$

$$= \pi - \left(\frac{\pi}{2} - B + \frac{\pi}{2} - A\right) = A + B = \pi - C;$$

therefore the radius of the circle circumscribing APB

$$= \frac{AB}{2 \sin APB} = \frac{AB}{2 \sin C} = R$$

the radius of the circle circumscribing the triangle ABC. Similarly, the radii of the circles circumscribing the triangles APC, BPC are each of them equal to R.

7. Let $\angle PAB$ (fig. 83) $= \theta$; $\angle APB = (n + 1)\,\alpha$;

$$\angle APC = \alpha; \quad \therefore\ \angle PCB = \theta + \alpha;$$

and if a be the radius of the circle,

$$AP = 2a \sin ABP = 2a \sin\{\theta + (n + 1)\,\alpha\};$$

hence the co-ordinates of P are

$$2a \sin\{\theta + (n + 1)\,\alpha\} \cos\theta; \quad 2a \sin\{\theta + (n + 1)\,\alpha\} \sin\theta;$$

and the equation to PC is

$$y - 2a \sin \{\theta + (n+1)\alpha\} \sin \theta$$
$$= \tan(\theta + \alpha) [x - 2a \sin\{\theta + (n+1)\alpha\} \cos\theta];$$

or $y\cos(\theta+\alpha) + 2a \sin\{\theta + (n+1)\alpha\} \sin\alpha = \sin(\theta+\alpha)\, x$;

$$\text{hence } y\cos(\theta+\alpha) + 2a\sin\alpha \,.\, \cos n\alpha \sin(\theta+\alpha)$$
$$+ 2a \sin\alpha \sin n\alpha \cos(\theta+\alpha) = \sin(\theta+\alpha)\, x;$$

which is always satisfied by making

$$x = 2a\sin\alpha\cos n\alpha\,; \quad y = -2a\sin\alpha\sin n\alpha\,;$$

hence PC always passes through a fixed point whose co-ordinates are $2a\sin\alpha\cos n\alpha$, $-2a\sin\alpha\sin n\alpha$.

8. From the vertex A (fig. 114) draw AB perpendicular to the base of the triangle; let C be the centre of one of the ellipses which touches the base in B, and one of the sides in P; S its focus; draw PN perpendicular to AB; then if

$$\angle PAB = \alpha, \;\; CN = y, \;\; PN = x, \;\; AC = X, \;\; CS = Y;$$

we have

$$X + b = AB = c\,; \quad AC = \frac{b^2}{y} = X; \quad \tan\angle PAB = \frac{a^2 y}{b^2 x} = \tan\alpha\,;$$

$$Y^2 = CS^2 = a^2 - b^2; \quad \frac{x}{a} = \frac{a\cot\alpha}{b} \cdot \left(\frac{y}{b}\right);$$

$$\frac{y^2}{b^2}\left(1 + \frac{a^2\cot^2\alpha}{b^2}\right) = 1, \text{ or } \frac{y^2}{b^4}(b^2 + a^2\cot^2\alpha) = 1\,;$$

$$\therefore\; X^2 = \frac{b^4}{y^2} = b^2 + a^2\cot^2\alpha\,; \quad Y^2 = a^2 - b^2;$$

$$\text{or } X^2 - Y^2\cot^2\alpha = b^2\operatorname{cosec}^2\alpha = (c - X)^2\operatorname{cosec}^2\alpha\,;$$

$$\therefore\; X^2\sin^2\alpha - Y^2\cos^2\alpha = c^2 - 2cX + X^2;$$

$$\text{hence } (X^2 + Y^2)\cos^2\alpha - 2cX + c^2 = 0\,;$$

$$\text{or } (X - c\sec^2\alpha)^2 + Y^2 = (c\sec\alpha\tan\alpha)^2,$$

which is the equation to a circle the co-ordinates of whose centre are $c\sec^2\alpha$, 0; and radius $c\sec\alpha\tan\alpha$.

If EF be drawn perpendicular to AE meeting AB in F; the circle described with centre F and radius FE will be the locus of S.

9. Since PQ (fig. 115) is parallel to AB, $\dfrac{BX}{QB} = \dfrac{AX}{AP}$.
From A, B draw AT, BT' tangents to the circle; then

$$AX.AP = AT^2; \quad BX.BQ = BT'^2;$$

$$\text{or } \frac{BX^2}{BT'^2} = \frac{AX^2}{AT^2}; \quad \therefore\ BX : AX :: BT' : AT.$$

Hence divide AB in D so that

$$AD : DB :: AT : BT' :: AX : BX;$$

$$\text{and } AE : EB :: AT : BT' :: AX : BX;$$

then the angles AXB, BXP will be bisected by the straight lines XD, XE; therefore $\angle EXD$ is a right angle, and if a circle be described upon the diameter DE it will intersect the given circle in two points X, X' either of which will satisfy the conditions of the problem.

10. Let A_1, A_2, A_3, A_4 be the areas of the four triangular faces of the pyramid; V its volume; then if C be the centre of the inscribed sphere, the volume of the pyramid is equal to the sum of the volumes of the four pyramids whose common vertex is C and bases A_1, A_2, A_3, A_4 respectively;

$$\therefore\ V = \frac{r}{3}(A_1 + A_2 + A_3 + A_4).$$

Also if C_1 be the centre of the sphere touching A_1, and the planes A_2, A_3, A_4 produced; the three pyramids whose bases are A_2, A_3, A_4 and vertex C_1, will exceed the original pyramid by the pyramid whose vertex is C_1 and base A_1; or

$$V = \frac{r_1}{3}(A_2 + A_3 + A_4 - A_1);$$

$$\text{similarly, } V = \frac{r_2}{3}(A_1 + A_3 + A_4 - A_2);$$

$$V = \frac{r_3}{3}(A_1 + A_2 + A_4 - A_3);$$

$$V = \frac{r_4}{3}(A_1 + A_2 + A_3 - A_4);$$

$$\therefore \frac{V}{r_1} + \frac{V}{r_2} + \frac{V}{r_3} + \frac{V}{r_4} = \tfrac{2}{3}(A_1 + A_2 + A_3 + A_4) = \frac{2V}{r};$$

$$\text{hence } \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{r}.$$

11. Let a, b; ma, mb be the semi-axes of the interior and exterior ellipses respectively; then the equation to a tangent at a point x, y of the interior ellipse is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1 \ldots (1).$$

Again, let X, Y be the co-ordinates of R; then if tangents be drawn from the point X, Y to the exterior ellipse, the equation to the chord of contact is

$$\frac{x'X}{m^2a^2} + \frac{y'Y}{m^2b^2} = 1;$$

and in order that this may coincide with equation (1) we have

$$x = \frac{X}{m^2}, \; y = \frac{Y}{m^2}; \text{ and } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1;$$

$$\therefore \left(\frac{X}{am^2}\right)^2 + \left(\frac{Y}{m^2b}\right)^2 = 1,$$

which is the equation to an ellipse whose semi-axes are m^2a, m^2b, and therefore similar to the two given ellipses.

12. Let $CP = a'$, $CQ = b'$; (fig. 116) $\angle CPQ = \phi$;
$\angle PCA = \theta$; $\angle QCa = \theta'$; $\angle PCQ = \pi - \alpha$;
X, Y the co-ordinates of R referred to the axes of the ellipse as axes; then

$$X = a'\cos\theta - PQ\cos(60 + \phi - \theta);$$
$$Y = a'\sin\theta + PQ\sin(60 + \phi - \theta);$$

$$\text{also } PQ\cos\phi = a' + b'\cos\alpha; \quad PQ\sin\phi = b'\sin\alpha;$$

$$\therefore X = a'\cos\theta - (a' + b'\cos\alpha)\cos(60-\theta) + b'\sin\alpha\sin(60-\theta)$$

$$= a'\left\{\cos\theta - \cos(60-\theta)\right\} - b'\cos(60-\theta+\alpha)$$

$$= a'\sin(30-\theta) - b'\sin(30-\theta');$$

$$Y = a'\sin\theta + (a' + b'\cos\alpha)\sin(60-\theta) + b'\sin\alpha\cos(60-\theta)$$

$$= a'\left\{\sin\theta + \sin(60-\theta)\right\} + b'\sin(60-\theta+\alpha)$$

$$= a'\cos(30-\theta) + b'\cos(30-\theta');$$

$$\text{now } a'\cos\theta = x,\ a'\sin\theta = y;$$

$$b'\cos\theta' = x' = \frac{a}{b}y; \quad b'\sin\theta' = \frac{b}{a}x;$$

$$\therefore X = \frac{x}{2} - \frac{\sqrt{3}}{2}y - \frac{a}{2b}y + \frac{b}{2a}\sqrt{3}\,.\,x = \frac{a+b\sqrt{3}}{2}\left(\frac{x}{a} - \frac{y}{b}\right);$$

$$Y = \frac{\sqrt{3}}{2}x + \frac{y}{2} + \frac{\sqrt{3}}{2}\frac{a}{b}y + \frac{b}{2a}x = \frac{b+a\sqrt{3}}{2}\left(\frac{x}{a} + \frac{y}{b}\right);$$

$$\text{or } \left(\frac{2X}{a+b\sqrt{3}}\right)^2 + \left(\frac{2Y}{b+a\sqrt{3}}\right)^2 = 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 2;$$

which is the equation to an ellipse whose semi-axes are

$$\frac{a+b\sqrt{3}}{\sqrt{2}}; \quad \frac{b+a\sqrt{3}}{\sqrt{2}}.$$

13. Let S' (fig. 117) be the focus of the conjugate hyperbola; $S'PK$ a normal; x, y the co-ordinates of P; then the equation to the normal $S'PK$ is

$$y' - y = \frac{-a^2y}{b^2x}(x' - x);$$

and when $x' = 0$,

$$y' = \left(1 + \frac{a^2}{b^2}\right)y = CS' = \sqrt{a^2+b^2};$$

$$\therefore y = \frac{b^2}{\sqrt{a^2+b^2}};$$

also when $y' = 0$, $x' = CK = \left(1 + \frac{b^2}{a^2}\right) x$;

and $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} = 1 + \frac{b^2}{a^2 + b^2} = \frac{a^2 + 2b^2}{a^2 + b^2}$;

$$\therefore\ x' = \frac{\sqrt{(a^2 + b^2)(a^2 + 2b^2)}}{a};$$

and the area included between the four normals

$$= 4 \triangle CS'K = 2x'y' = \frac{2(a^2 + b^2)}{a}\sqrt{a^2 + 2b^2}.$$

14. Let PQ, PQ' (fig. 118) be taken for the co-ordinate axes; $PQ = a$, $PQ' = b$; then the equation to the parabola is

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1;$$

and the equation to QQ' is $\frac{x}{a} + \frac{y}{b} = 1$; also let $y = mx$ be the equation to $PABC$, x', x'', x''' the co-ordinates of A, B, C respectively; hence

$$\sqrt{\frac{x'}{a}} + \sqrt{\frac{mx'}{b}} = 1, \quad \sqrt{\frac{x'''}{a}} - \sqrt{\frac{mx'''}{b}} = 1, \quad \frac{x''}{a} + \frac{mx''}{b} = 1;$$

$$\text{or}\quad \frac{1}{x'} + \frac{1}{x'''} = 2\left(\frac{1}{a} + \frac{m}{b}\right) = \frac{2}{x''}.$$

But PA, PB, PC are proportional to x', x'', x''';

$$\therefore\ \frac{1}{PA} + \frac{1}{PC} = \frac{2}{PB};$$

and $PABC$ is divided harmonically.

15. Let CA, CA' (fig. 119) be the major-axes of the two ellipses inclined at an angle α; their polar equations referred to the fixed line CA are

$$\rho^2 = \frac{b^2}{1 - e^2\cos^2\theta}, \quad \rho'^2 = \frac{b^2}{1 - e^2\cos^2(\theta - \alpha)};$$

hence at the points of intersection

$$\cos\theta = \pm\cos(\theta - \alpha);$$

$$\therefore\ \theta = \frac{\alpha}{2},\quad \frac{\pi}{2} + \frac{\alpha}{2},\quad \pi + \frac{\alpha}{2},\quad \frac{3\pi}{2} + \frac{\alpha}{2}.$$

When $\angle ACP = \frac{\alpha}{2}$, produce CP to Q, and draw PT a tangent at P; then the angle at which the curves intersect $= 2\angle TPQ$;

$$\text{but}\ \cot TPQ = \frac{e^2\sin\theta\cos\theta}{1 - e^2\cos^2\theta}\ \text{where}\ \theta = \frac{\alpha}{2};$$

$$\therefore\ 2\angle TPQ = 2\cot^{-1}\frac{e^2\sin\alpha}{2\left(1 - e^2\cos^2\frac{\alpha}{2}\right)}.$$

At the second point of intersection $\theta = \frac{\pi}{2} + \frac{\alpha}{2}$; therefore the angle between the curves

$$= 2\cot^{-1}\frac{e^2\sin\alpha}{2\left(1 - e^2\sin^2\frac{\alpha}{2}\right)}.$$

16. Let AB (fig. 120) be the slant side of the cone; BC the diameter of the base; BD the axis-major of the elliptic section; draw DE parallel to BC, and AP perpendicular to BD; then the axis-minor of the elliptic section

$$= \sqrt{CB\,.\,DE},\quad \angle CAB = 2\alpha,\quad \angle ABD = \theta;$$

$$\text{and}\ \frac{\text{volume of oblique section of the cone}}{\text{volume of the cone}}$$

$$= \frac{\frac{1}{3}\,.\,AP \times \text{area of elliptic section}}{\frac{1}{3}\,.\,AB\cos\alpha \times \text{area of circular base}};$$

$$\text{or}\quad \frac{1}{n} = \frac{AB\sin\theta \times \pi\,.\,BD\,.\sqrt{CB\,.\,DE}}{AB\cos\alpha \times \pi\,.\,BC^2}$$

$$= \frac{\sin\theta}{\cos\alpha}\cdot\frac{BD}{BC}\cdot\sqrt{\frac{DE}{BC}} = \frac{\sin\theta}{\cos\alpha}\cdot\frac{BD}{BC}\sqrt{\frac{AD}{AB}};$$

$$\text{hence } \frac{1}{n} = \frac{\sin\theta}{\cos\alpha} \cdot \frac{\cos\alpha}{\sin(2\alpha+\theta)} \sqrt{\frac{\sin\theta}{\sin(2\alpha+\theta)}}$$

$$= \left\{\frac{\sin\theta}{\sin(2\alpha+\theta)}\right\}^{\frac{3}{2}};$$

$$\therefore\ n^{\frac{2}{3}} = \frac{\sin(2\alpha+\theta)}{\sin\theta} = \sin 2\alpha \cot\theta + \cos 2\alpha.$$

17. See Appendix, Art. 75.

18. (1) $y^2 - (4x+2)\,y + (2x+1)^2 + x^2 - 4x + 4 = 0$;

$$\therefore\ (y - 2x + 1)^2 + (x-2)^2 = 0;$$

which equation can only be satisfied by making $x - 2 = 0$, and $y - 2x - 1 = 0$; $\therefore\ x = 2,\ y = 5$; or the locus is a point whose co-ordinates are 2, 5.

(2) Transform the origin to a point α, β in the curve, by making $x = x' + \alpha$, $y = y' + \beta$;

$$\therefore\ (y'+\beta)^2 - 2(x'+\alpha)(y'+\beta) + (x'+\alpha)^2 - 2(y'+\beta) - 2(x'+\alpha) - 3 = 0;$$

$$\text{or } (y'-x')^2 + (2\beta - 2\alpha - 2)\,y' + (2\alpha - 2\beta - 2)\,x' = 0,$$

$$\text{and } (\beta-\alpha)^2 - 2\beta - 2\alpha - 3 = 0.$$

Let $\dfrac{\beta - \alpha - 1}{\alpha - \beta - 1} = 1 = \tan\delta$; and transform the equation to polar co-ordinates;

$$\therefore\ 2\rho\sin^2(\theta-\delta) + 2\sqrt{2}\,(\beta-\alpha-1)\cos(\theta-\delta) = 0;$$

$$\text{hence } \alpha = \beta = -\tfrac{3}{4},\quad \rho\sin^2(\theta-\delta) - \sqrt{2}\cos(\theta-\delta) = 0;$$

which is the equation to a parabola, the co-ordinates of whose vertex are $\alpha = \beta = -\frac{3}{4}$; the latus rectum $= \sqrt{2}$; and axis inclined to the axis of x at an angle $\delta = 45$. See fig. 121.

19. See Appendix, Art. 31.

ST JOHN'S COLLEGE. Dec. 1843. (No. XIV.)

1. Similar triangles are to one another in the duplicate ratio of their homologous sides.

2. Draw a straight line perpendicular to a plane from a given point without it.

3. Shew that the equation to the ellipse referred to any pair of tangents as axes, is

$$\left(\frac{y-k}{k}\right)^2 + \left(\frac{x-h}{h}\right)^2 + cxy = 1,$$

h, k being the portions of the tangents intercepted between the ellipse and their common point of intersection.

Find the corresponding equation to the parabola.

4. Prove analytically that if from a point without a circle two straight lines be drawn, one of which cuts the circle and the other touches it, the square of the line that touches the circle is equal to the rectangle contained by the straight line which cuts the circle and the part of it without the circle.

5. The portions of the tangent at any point of an equiangular hyperbola intercepted between the curve and asymptote, are equal to the distance of the same point from the centre.

6. Find the locus of the middle points of all chords of an ellipse, (1) which pass through the extremity of the axis-major; (2) which pass through the focus.

7. Find the diameter of the circle whose equation is

$$x^2 + y^2 + 2xy\cos\omega = ax + by.$$

8. A straight line of given length slides between a circle and a straight line; find the locus of the middle point.

9. Two straight lines revolve uniformly in one plane about one extremity; the one moving twice as fast as the other. Find the locus of their points of intersection, supposing them to begin to move together from the straight line joining their fixed extremities.

10. Find a point external to two circles in the same plane that do not meet, such that if straight lines be drawn through it cutting both circles, the portions of all such straight lines intercepted within the circles shall be proportional to their radii.

11. Hence draw a pair of common tangents to two circles; and determine within what limits a point must be situated, so that a straight line may be drawn from it cutting both.

12. Two straight lines include a given angle 2α, and from their point of intersection a straight line of given length is drawn bisecting the angle between them. Determine the locus of the middle points of all straight lines drawn through the extremity of this line to meet the other two.

13. Lines are drawn through the angular points A, B, C of a triangle through any point meeting the opposite sides in a, b, c; a circle is described through these three points cutting the same sides in a', b', c'; shew that Aa', Bb', Cc' meet in one point. Assuming that $Ab \,.\, Bc \,.\, Ca = Ba \,.\, Cb \,.\, Ac$ and conversely, that when this relation holds the lines pass through one point.

14. If the angle between the focal distances of a conic section be constant and $= 2\alpha$, the locus of the intersection of the tangents at their extremities has for its polar equation

$$\rho\,(\cos\alpha + e\cos\theta) = a\,(1 - e^2).$$

15. Find the locus of a point from which if perpendiculars be dropped on three given straight lines their points of intersection shall all lie in a straight line.

16. In an ellipse if two focal distances r and r' include an angle 2α, and T be the intersection of the tangents at their extremities; then (1) the angle between the focal distances is bisected by ST, and (2) $ST^2 = \dfrac{b^2 rr'}{b^2 - rr' \sin^2 \alpha}$.

Hence shew that for the parabola $ST^2 = rr'$ always; but in the ellipse only when $\alpha = 0$.

17. Find the locus of the centres of all circles which cut off from the directions of two sides of a triangle chords equal to two given straight lines.

Hence describe a circle that shall cut off from the direction of three sides of a triangle chords respectively equal to three given straight lines.

SOLUTIONS TO (No. XIV.)

1. EUCLID, Prop. 19. Book VI.

2. Euclid, Prop. 11. Book XI.

3. Hymers' Conic Sections, Art. 228.

4. Let S (fig. 122) be the given point without the circle; C the centre; P any point in the circle; $SC = c$; $CP = a$; $SP = \rho$; $\angle PSC = \theta$;

$$\therefore\ a^2 = \rho^2 + c^2 - 2c\rho\cos\theta, \text{ or } \rho^2 - 2c\cos\theta\rho + c^2 - a^2 = 0;$$

and if SP be produced to meet the circle in P', the two values of ρ are SP, SP'; hence $SP \times SP' = c^2 - a^2$, and is independent of θ; but when SP is moved until P and P' coincide in T, SP and SP' become equal ST, and ST becomes a tangent; hence $ST^2 = c^2 - a^2$; or $SP \,.\, SP' = ST^2$.

5. Let TPt (fig. 123) be a tangent to the hyperbola at P, meeting the asymptotes in T, t; then $TP = Pt$; but $PT = CD$; and in the equiangular hyperbola

$$CD^2 - CP^2 = BC^2 - AC^2 = 0;$$

$$\therefore\ CD = CP; \text{ hence } TP = Pt = CP.$$

6. (a) Let A (fig. 124) be the extremity of the axis-major; P any point in the ellipse whose centre is C; Q the middle point of AP; $\angle PAC = \theta$; $AP = \rho$; $AQ = \rho'$;

$$\therefore\ \rho^2\sin^2\theta = \frac{b^2}{a^2}(2a\rho\cos\theta - \rho^2\cos^2\theta),$$

$$\text{or } \rho(1 - e^2\cos^2\theta) = \frac{2b^2}{a}\cos\theta; \text{ and } \rho = 2\rho';$$

$$\therefore\ \rho'(1 - e^2\cos^2\theta) = \frac{b^2}{a}\cos\theta;$$

which is the equation to a similar ellipse whose semi-axes are

$$\frac{a}{2},\ \frac{b}{2}.$$

(β) Let PSp (fig. 125) be any chord passing through the focus; Q its middle point; $SQ = r$, $\angle PSC = \theta$; then

$$\frac{SP - Sp}{2} = \tfrac{1}{2}\left\{\frac{a(1-e^2)}{1 - e\cos\theta} - \frac{a(1-e^2)}{1 + e\cos\theta}\right\},$$

$$\text{or } r = \frac{a(1-e^2)\,e\cos\theta}{1 - e^2\cos^2\theta};$$

which is the equation to an ellipse whose vertex is S, eccentricity e, and axis-major $2ae$.

7. The equation to a circle referred to oblique axes inclined at an angle ω is

$$(x-\alpha)^2 + (y-\beta)^2 + 2(x-\alpha)(y-\beta)\cos\omega = r^2;$$

and comparing this with the given equation, we have

$$2(\alpha + \beta\cos\omega) = a; \quad 2(\beta + \alpha\cos\omega) = b;$$

$$\alpha^2 + \beta^2 + 2\alpha\beta\cos\omega = r^2;$$

$$\therefore\ \alpha = \frac{a - b\cos\omega}{2\sin^2\omega}; \quad \beta = \frac{b - a\cos\omega}{2\sin^2\omega},$$

$$\text{and } r^2 = \frac{a\alpha + b\beta}{2} = \frac{a^2 - 2ab\cos\omega + b^2}{4\sin^2\omega};$$

whence the radius, and co-ordinates of the centre are determined.

Any line passing through the centre will be a diameter to the circle, and $y - \beta = m(x - \alpha)$ will be its equation where m is arbitrary.

To determine the diameter which passes through the origin, its equation is

$$y = \frac{\beta}{\alpha}x, \text{ or } y = \frac{b - a\cos\omega}{a - b\cos\omega}x.$$

8. Let C (fig. 126) be the centre of the circle; draw CA perpendicular to the given straight line AB; and let AC, AB be taken for the co-ordinate axes; suppose PQ any position of

the line whose length is l; draw PM perpendicular to AC, and let $AM = x$, $MP = y$, $AQ = y'$; $AC = c$, $CP = r$;

$$\therefore\ x^2 + (y - y')^2 = l^2,\ y^2 + (c - x)^2 = r^2;$$

and if X, Y be the co-ordinates of the middle point of PQ,

$$X = \frac{x}{2};\ Y = \frac{y + y'}{2};\ \therefore\ y - y' = \sqrt{l^2 - 4X^2};\ y + y' = 2Y;$$

$$\text{or } 2y = 2Y + \sqrt{l^2 - 4X^2} = 2\sqrt{r^2 - (c - 2X)^2},$$

which is the equation required.

9. (1) Let A, B (fig. 127) be the two fixed points; P the intersection of the lines PA, PB in any position; then if

$$\angle PAB = \theta, \text{ and } \angle PBC = 2\theta,$$

$$\text{we have } \angle APB = \theta, \text{ and } PB = AB;$$

hence the locus of P is a circle whose centre is B.

(2) If $\angle ABP = 2\theta$, $AP = \rho$, $AB = a$, then

$$\rho = \frac{a \sin 2\theta}{\sin 3\theta} = \frac{2a \cos\theta}{3 - 4\sin^2\theta},\ \text{or } \rho = \frac{2a\cos\theta}{4\cos^2\theta - 1};$$

$$\therefore\ 4x^2 - (x^2 + y^2) = 2ax,\ \text{or } 3x^2 - y^2 = 2ax;$$

$$\text{hence } y^2 = 3\left\{\left(x - \frac{a}{3}\right)^2 - \frac{a^2}{9}\right\},$$

which is the equation to a hyperbola whose semi-axes are $\frac{a}{3}$, $\frac{a}{\sqrt{3}}$, and eccentricity 2.

B is the focus of the hyperbola, and A the vertex of the exterior hyperbola.

10. Let $PQRS$ (fig. 128) be a straight line cutting the two circles whose centres are A, B, and intercepting the chords PQ, RS respectively proportional to AP, BR. Produce BA to meet $SRQP$ produced in T; and draw AM, BN perpendicular to PS; then

$$\frac{PQ}{AP} = \frac{RS}{BR},\ \text{or } \frac{MP}{AP} = \frac{RN}{BR};$$

therefore AP is parallel to BR, and

$$\frac{AT}{BT}=\frac{AP}{BR}, \therefore \frac{AT}{AB}=\frac{AP}{BR-AP};$$

hence AT is constant; and every line passing through the fixed point T and cutting both circles will intercept chords PQ, RS proportional to their radii.

If T' be between the circles, it may be proved in like manner, that $\dfrac{AT'}{AB}=\dfrac{AP}{BR+AP}$; and every line drawn through T' will intercept chords proportional to the radii of the circles.

11. Since PQ, RS are always proportional to the radii of the circles, they vanish together; or when P and Q coincide in P', R and S will coincide in R', and $TP'R'$ will be a common tangent; also $\angle AP'T$ is a right angle; hence if AT be taken a fourth proportional to $BR-AP$, AP and AB, and upon the diameter AT a circle be described cutting the given circle in P', P''; TP', TP'' will be common tangents.

Similarly, if AT' be taken a fourth proportional to $BR+AP$, AP and AB, and a circle be described on AT', this will meet the given circle in two points Q', Q'' and $T'Q'$, $T'Q''$ will be common tangents.

Also let $DT'E'$, $D'T'E$ be the pair of common tangents; then if any point be taken within the angles $DT'E$, or $D'T'E'$ and exterior to the bases DE, $D'E'$ it will be impossible to draw a line cutting both circles, but a straight line may be drawn through a point in any other position so as to cut both circles.

12. Let ASB (fig. 129) be the line bisecting the given angle; $AS=a$, PSp any line passing through S; Q the middle point of Pp; $\angle PAS=\alpha$; $\angle PSB=\theta$; $SQ=\rho$; then

$$SP=\frac{a\sin\alpha}{\sin(\theta-\alpha)}, \quad Sp=\frac{a\sin\alpha}{\sin(\theta+\alpha)};$$

$$\therefore \; SQ = \rho = \tfrac{1}{2}(SP - Sp) = \frac{a \sin \alpha}{2}\left\{\frac{1}{\sin(\theta - \alpha)} - \frac{1}{\sin(\theta + \alpha)}\right\}$$

$$= \frac{a \sin^2 \alpha \,.\, \cos\theta}{\sin^2\theta - \sin^2\alpha};$$

$$\therefore \; y^2 - \sin^2\alpha\,(x^2 + y^2) = a \sin^2\alpha \; x,$$

$$\text{or } y^2 \cos^2\alpha - x^2 \sin^2\alpha = a \sin^2\alpha \; x;$$

$$\therefore \; \frac{\left(x + \dfrac{a}{2}\right)^2}{\left(\dfrac{a}{2}\right)^2} - \frac{y^2}{\left(\dfrac{a}{2}\tan\alpha\right)^2} = 1,$$

which is the equation to a hyperbola whose axes are a, $a \tan \alpha$, and whose centre is the middle point of AS.

13. Fig. 130. $\quad Ab \,.\, Ab' = Ac \,.\, Ac'$;

$$Bc \,.\, Bc' = Ba \,.\, Ba';$$

$$Ca \,.\, Ca' = Cb \,.\, Cb';$$

$$\therefore \; (Ab . Bc . Ca.)(Ab' . Bc' . Ca') = (Ac . Ba . Cb.)\, Ac' . Ba' . Cb';$$

$$\text{but } Ab \,.\, Bc \,.\, Ca = Ac \,.\, Ba \,.\, Cb;$$

$$\therefore \; Ab' \,.\, Bc' \,.\, Ca' = Ac' \,.\, Ba' \,.\, Cb';$$

or Aa', Bb', Cc' meet in the same point.

14. (a) Let C be the centre, (fig. 131) S the focus, PT, QT two tangents meeting in T;

$$\angle PSC = \theta; \;\; \angle PSQ = 2\alpha; \;\; \therefore \; \angle PST = \alpha;$$

$$\angle SPT = \phi; \;\; \angle TSC = \psi; \;\; ST = \rho;$$

$$\therefore \; \rho = \frac{SP \,.\, \sin\phi}{\sin(\phi + \alpha)} = \frac{SP}{\cos\alpha + \sin\alpha \cot\phi};$$

$$\text{and } \cot\phi = \frac{e \sin\theta}{1 - e\cos\theta}; \quad SP = \frac{a(1 - e^2)}{1 - e\cos\theta};$$

$$\therefore\ \rho = \frac{a\,(1-e^2)}{(1-e\cos\theta)\cos\alpha + e\sin\theta\sin\alpha},$$

$$\text{or}\ \ \rho = \frac{a\,(1-e^2)}{\cos\alpha - e\cos(\theta+\alpha)} = \frac{a\,(1-e^2)}{\cos\alpha - e\cos\psi}.$$

$$(\beta)\ \text{Also}\ \ \frac{1}{\rho^2} = \frac{\cos^2\alpha - 2e\cos\alpha\cos(\theta+\alpha) + e^2\cos^2(\theta+\alpha)}{\{a\,(1-e^2)\}^2},$$

and if $SP = r$, $SQ = r'$; $\dfrac{1}{r} = \dfrac{1-e\cos\theta}{a\,(1-e^2)}$; $\dfrac{1}{r'} = \dfrac{1-e\cos(\theta+2\alpha)}{a\,(1-e^2)}$;

$$\therefore\ \frac{1}{rr'} = \frac{1 - e\,\{\cos\theta + \cos(\theta+2\alpha)\} + e^2\,\{\cos^2(\theta+\alpha) - \sin^2\alpha\}}{\{a\,(1-e^2)\}^2};$$

$$\text{hence}\ \ \frac{1}{\rho^2} = \frac{1}{rr'} - \frac{\sin^2\alpha\,(1-e^2)}{a^2\,(1-e^2)^2}$$

$$= \frac{1}{rr'} - \frac{\sin^2\alpha}{b^2};\ \ \text{or}\ \rho^2 = \frac{b^2 rr'}{b^2 - rr'\sin^2\alpha};$$

which shews that in the ellipse ρ^2 cannot $= rr'$ unless $\alpha = 0$; but in the parabola b is infinite, $\therefore\ \rho^2 = rr'$.

15. Let the base AB (fig. 132) be taken for the axis of x; A the origin; x, y the co-ordinates of the point P from which the perpendiculars Pa, Pb, Pc are drawn upon the three sides of the triangle ABC; then the co-ordinates of the points c, b, a respectively are x, 0;

$$(x\cos A + y\sin A)\cos A,\ \ (x\cos A + y\sin A)\sin A\,;$$

$$c - \{(c-x)\cos B + y\sin B\}\cos B;\ \ \{(c-x)\cos B + y\sin B\}\sin B;$$

but when three points x_1, y_1; x_2, y_2; x_3, y_3 are in the same straight line,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1};$$

hence if a, b, c be in the same straight line,

$$\frac{(x\cos A+y\sin A)\sin A-0}{(x\cos A+y\sin A)\cos A-x}=\frac{\{(c-x)\cos B+y\sin B\}\sin B-0}{c-\{(c-x)\cos B+y\sin B\}\cos B-x},$$

$$\text{or } \frac{x\cos A+y\sin A}{y\cos A-x\sin A}=\frac{(c-x)\cos B+y\sin B}{(c-x)\sin B-y\cos B},$$

$$\therefore x(c-x)(\sin A\cos B+\cos A\sin B)-y(c-x)(\cos A\cos B-\sin A\sin B)$$
$$-y^2(\sin A\cos B+\cos A\sin B)-xy(\cos A\cos B-\sin A\sin B)=0;$$

$$\therefore (cx-x^2)\sin(A+B)-cy(\cos A+B)-y^2\sin(A+B)=0;$$

$$\text{or } cx-x^2-cy\cot(A+B)-y^2=0;$$

$$\text{hence } x^2+y^2-cx-cy\cot C=0;$$

$$\therefore \left(x-\frac{c}{2}\right)^2+\left(y-\frac{c}{2}\cot C\right)^2=\left(\frac{c}{2}\operatorname{cosec} C\right)^2;$$

which is the equation to the circle circumscribing the triangle.

16. See No. 14. (β)

17. Let O (fig. 133) be the centre and r the radius of the circle which intercepts from AB, AC the two portions cc', $bb'=\gamma$, β respectively; draw OM parallel to AC; and let

$$AM=x,\ MO=y;\quad \therefore r^2-y^2\sin^2 A=\frac{\gamma^2}{4}.$$

$$\text{Similarly,}\quad r^2-x^2\sin^2 A=\frac{\beta^2}{4};$$

$$\therefore x^2-y^2=\frac{\gamma^2-\beta^2}{4\sin^2 A};$$

or the locus of O is an equilateral hyperbola whose centre is A, having two conjugate diameters in the directions AB, AC.

Again, let ON be drawn parallel to BC, and let $BN=x'$, $ON=y'$; then if $aa'=\alpha$,

$$x'^2-y'^2=\frac{\gamma^2-\alpha^2}{4\sin^2 B};$$

and if two hyperbolas be drawn with centres A, B, whose equations are

$$x^2 - y^2 = \frac{\gamma^2 - \beta^2}{4 \sin^2 A}; \quad x'^2 - y'^2 = \frac{\gamma^2 - \alpha^2}{4 \sin^2 B},$$

the intersection of the two hyperbolas will give the centre of the circle.

There will be four points, viz. one within the triangle, and another between each side and the two remaining sides produced.

ST JOHN'S COLLEGE. Dec. 1844. (No. XV.)

1. If two parallel planes be cut by another plane, their common sections with it are parallel.

Perpendiculars are drawn from a point to a plane, and to a straight line in that plane; shew that the line joining the feet of the perpendiculars is perpendicular to the former line.

2. The sides containing a given angle are in a given ratio, and the vertex is fixed; shew that if the extremity of one of the sides moves in a given line, so does the extremity of the other.

3. A series of triangles are constructed on a given base, their vertices being in a line parallel to the base; shew that the perpendiculars through the extremties of the base to the sides of these triangles will intersect in a parabola whose latus rectum is the distance between the lines.

4. CP, CD are a pair of semi-conjugate diameters in an ellipse whose foci are S, H (S being the nearer to P) prove the following properties:

(α) If the ordinates at P and D be produced to meet the circumscribing circle in Q and E; then QCE is a right angle.

(β) The normals at P and D meet in the line through C perpendicular to PD.

(γ) The sum of the squares of the perpendiculars from P and D on any fixed diameter is constant.

(δ) The perpendiculars from P and D on diameters drawn respectively parallel to SD and HP are each equal to the semi-axis minor.

5. Straight lines are drawn from the extremities of a given diameter of a circle whose radius is (a) to the extremities of a chord which always subtends an angle α at the centre;

shew that their intersection traces out a circle whose radius $= a \sec \frac{\alpha}{2}$, whether the diameter and chord are joined towards the same or opposite parts.

6. If a circle cut a conic section the chords joining the points of intersection are equally inclined to the axis.

7. If a triangle formed by a pair of tangents to a conic section and the chord of contact be of constant area, the vertex traces out a similar and similarly situated conic section.

8. Two focal chords of a conic section are drawn, and the lines joining their extremities are produced to meet in two points P, Q; shew that SP is a perpendicular to SQ: and if the focal chords be produced to meet PQ, each of them as well as PQ will be harmonically divided.

9. If a series of chords to an ellipse pass through a fixed point, so do the chords of the corresponding conjugate arcs.

10. A pair of tangents are drawn at the points P, Q of a conic section, and another tangent RST meeting them respectively in R and T; shew that for every position of this latter tangent the ratio $\frac{RP}{RS} : \frac{TQ}{TS}$ is given; and RT subtends a constant angle at the focus.

11. Find the axes of the curve $y^2 + xy + x^2 = 1$, the angle between the co-ordinate axes being 45^0.

SOLUTIONS TO (No. XV.)

1. EUCLID, Prop. 16, Book XI.

Let A (fig. 134) be the given point, AB a perpendicular to the plane, AC perpendicular to CD in the plane; join CB, and draw CE parallel to AB; then since AB is perpendicular to the plane BCD, CE is perpendicular to the same plane; therefore CE is perpendicular to CD, or CD is perpendicular to CE and CA, and therefore perpendicular to the plane ACB; hence CD is perpendicular to CB.

2. Let A (fig. 135) be the fixed vertex; BC the straight line along which the extremity of one of the sides moves; draw AB perpendicular to BC; take any point M in BC; join AM, and make $\angle MAP = \alpha$, and $AP = n\,(AM)$; let

$$AB = a, \quad AP = \rho, \quad \angle BAP = \theta;$$

$$\therefore\ AM = a \sec(\theta - \alpha), \text{ and } AP = \rho = na \sec(\theta - \alpha);$$

hence the locus of P is a straight line making an angle $90 + \alpha$ with AB, whose perpendicular distance from the origin is na.

3. Let AB (fig. 136) be the given base, C the vertex of one of the triangles; draw CM perpendicular to AB; and Aa perpendicular to CB, meeting CM in P; then P is the intersection of the perpendiculars Aa, Bb upon BC, AC respectively; let $AM = x$, $MP = y$, $CM = p$, which is constant;

$$\therefore\ y = x \tan PAM = x \cot B;$$

$$\text{and } CM = BM \,.\, \tan B, \text{ or } p = (c - x) \tan B;$$

$$\therefore\ py = cx - x^2, \text{ or } p\left(\frac{c^2}{4p} - y\right) = \left(x - \frac{c}{2}\right)^2,$$

which is the equation to a parabola the co-ordinates of whose vertex are $\frac{c}{2}$, $\frac{c^2}{4p}$ and latus rectum $= p$.

4. (α) Let QPM, EDN (fig. 137) be the ordinates through the points P, D; then

$$\sin QCM = \frac{QM}{CQ} = \frac{a}{b}\cdot\frac{PM}{CQ} = \frac{CN}{CE} = \cos ECN;$$

$$\therefore\ \angle ECN + \angle QCM = \frac{\pi}{2},\ \text{or}\ \angle ECQ = \frac{\pi}{2}.$$

(β) Let x, y; x', y' be the co-ordinates of P and D respectively; then the equation to the normal at P is

$$Y - y = \frac{a^2 y}{b^2 x}(X - x),$$

$$\text{or } Y = \frac{a^2 y}{b^2 x}X + \left(1 - \frac{a^2}{b^2}\right)y = -\frac{x'}{y'}X + \left(1 - \frac{a^2}{b^2}\right)y; \quad (1)$$

similarly, the equation to the normal at D is

$$Y = -\frac{x}{y}X + \left(1 - \frac{a^2}{b^2}\right)y'; \quad (2)$$

multiply (1) by y', and (2) by y and subtract; therefore at the point of intersection of the two normals

$$(y' - y)\,Y = -(x' - x)\,X;$$

but the equation to a line through C perpendicular to PD is

$$y_1 = -\frac{x' - x}{y' - y}x_1;$$

therefore X, Y are co-ordinates of a point in this line; or the normals at P and D intersect in the perpendicular drawn from C upon PD.

(γ) Let a diameter be drawn making an angle α with the axis-major, and let PM', DN' be perpendiculars upon it from P, D;

$$\therefore\ PM' = x\sin\alpha + y\cos\alpha,$$

$$DN' = y'\cos\alpha + x'\sin\alpha = \frac{bx}{a}\cos\alpha - \frac{ay}{b}\sin\alpha;$$

$$\text{hence } (PM')^2 + (DN')^2 = (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$
$$= a^2 \sin^2 \alpha + b^2 \cos^2 \alpha\,;$$

and is constant for every pair of conjugate diameters.

(δ) Let CR be drawn parallel to SD; then the perpendicular from P on $CR = CP \,.\, \sin PCR = CP \,.\, \sin SDT$

$$= CP \,.\, \sqrt{\frac{b^2}{SD \,.\, HD}} = CP\sqrt{\frac{b^2}{CP^2}} = b.$$

Similarly, the perpendicular from P on the diameter parallel to $HD = b$.

5. Let AB (fig. 138) be the diameter of the circle; PQ a chord subtending an angle α at the centre; join AP, BQ meeting in R, and AQ, BP meeting in R'; then

$$\angle ARB = \angle APB - \angle PBQ = \frac{\pi}{2} - \frac{\alpha}{2};$$

and is constant; hence the locus of R is a circle whose radius

$$= \frac{AB}{2 \sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)} = a \sec \frac{\alpha}{2}.$$

$$\text{Also } \angle AR'B = \angle AQB + \angle PBQ = \frac{\pi}{2} + \frac{\alpha}{2};$$

therefore the locus of R' is a circle whose radius

$$= \frac{AB}{2 \sin\left(\frac{\pi}{2} + \frac{\alpha}{2}\right)} = a \sec \frac{\alpha}{2}.$$

6. Let P, Q, R, S be the four points of intersection of the circle and conic section;

$$y - m_1 x - c_1 = 0\,; \qquad y - m_2 x - c_2 = 0\,;$$
$$y - m_3 x - c_3 = 0\,; \qquad y - m_4 x - c_4 = 0\,;$$

the equations to PQ, QR, RS, SP respectively; then the equation to the conic section is

$$(y - m_1 x - c_1)(y - m_3 x - c_3) + \lambda(y - m_2 x - c_2)(y - m_4 x - c_4) = 0; \quad (1)$$

and the equation to the circle

$$(y - m_1 x - c_1)(y - m_3 x - c_3) + \lambda'(y - m_2 x - c_2)(y - m_4 x - c_4) = 0; \quad (2)$$

where λ, λ' are constants.

Now when the conic section is referred to the axis-major, the coefficient of xy vanishes;

$$\therefore\ m_1 + m_3 + \lambda(m_2 + m_4) = 0,$$

and in the circle

$$m_1 + m_3 + \lambda'(m_2 + m_4) = 0;$$

$\therefore\ m_2 + m_4 = 0$, since λ and λ' are different constants; hence

$$m_4 = -m_2; \text{ and } m_1 + m_3 = 0, \text{ or } m_3 = -m_1;$$

hence PQ, RS are equally inclined to the axis, and also QR, SP.

If equation (2) represents any conic section whose axis is parallel to that represented by equation (1) the coefficient of xy will vanish in equation (2); and we shall still have

$$m_3 = -m_1, \quad m_4 = -m_2;$$

hence if two conic sections whose axes are parallel, intersect one another; the straight lines joining the points of intersection will be equally inclined to the axis. In like manner it may be proved that PS and QR are equally inclined to the axis.

7. Let Qq (fig. 139) be a chord of an ellipse whose centre is C; QT, qT two tangents meeting in T; join CT meeting the ellipse in P and Qq in V; let $CV = x$, $QV = y$, $CP = a'$; then

$$CT = \frac{a'^2}{x}, \quad TV = \frac{a'^2}{x} - x = \frac{a'^2}{b'^2} \cdot \frac{y^2}{x};$$

$$\text{and } \triangle QTq = QV \,.\, TV \sin TVQ$$

$$= y\left(\frac{a'^2}{b'^2} \cdot \frac{y^2}{x}\right) \cdot \left(\frac{ab}{a'b'}\right) = \frac{a'}{x}\left(\frac{y}{b'}\right)^3 ab = m^2;$$

$$\therefore \left\{1 - \left(\frac{x}{a'}\right)^2\right\}^{\frac{3}{2}} = \left(\frac{m^2}{ab}\right) \cdot \left(\frac{x}{a'}\right);$$

hence $\frac{x}{a'}$ is constant $= n$; and if $CT = \rho$, $\rho = \frac{a'^2}{x} = \frac{a'}{n}$; therefore the locus of T is an ellipse similar to that of P, and its semi-axes will be $\frac{a}{n}$, $\frac{b}{n}$ respectively; where

$$(1 - n^2)^3 = \frac{m^4}{a^2 b^2} \cdot n^2.$$

8. (a) Let RSr, $R'Sr'$ (fig. 140) be two chords drawn through the focus S of a conic section; take SR, SR' the axes of x and y respectively; let $SR = \alpha$, $SR' = \beta$; $Sr = \alpha'$, $Sr' = \beta'$: then the equations to rR', $r'R$ respectively become

$$\frac{y}{\beta} - \frac{x}{\alpha'} = 1; \text{ and } \frac{x}{\alpha} - \frac{y}{\beta'} = 1;$$

therefore by subtraction

$$\left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right)x - \left(\frac{1}{\beta} + \frac{1}{\beta'}\right)y = 0;$$

$$\text{but } \frac{1}{\alpha} + \frac{1}{\alpha'} = \frac{1}{\beta} + \frac{1}{\beta'}; \ \therefore x - y = 0,$$

and if rR' $r'R$ intersect in P, the equation to SP is $y - x = 0$.

Again, the equations to RR', rr' are

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1, \quad \frac{x}{\alpha'} + \frac{y}{\beta'} = -1;$$

therefore by addition

$$\left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right)x + \left(\frac{1}{\beta} + \frac{1}{\beta'}\right)y = 0;$$

therefore $x + y = 0$, which is the equation to SQ; hence SP and SQ are at right angles.

(β) If x', y' be the co-ordinates of P, and x'', y'' the co-ordinates of Q, then the equation to PQ is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'), \text{ but } y' = x', \; y'' = -x'';$$

$$\therefore \; y - x' = -\frac{x'' + x'}{x'' - x'}(x - x');$$

let this meet the axis of y in V;

$$\therefore \; SV = x'\left(1 + \frac{x'' + x'}{x'' - x'}\right) = \frac{2x'x''}{x'' - x'};$$

$$\therefore \; \frac{2}{SV} = \frac{1}{x'} - \frac{1}{x''} = \frac{1}{y'} + \frac{1}{y''}.$$

Also by combining the equations to rR', SP we have

$$\frac{1}{x'} = \frac{1}{\beta} - \frac{1}{\alpha};$$

and by combining the equations to rr', SQ we have

$$\frac{1}{x''} = \frac{1}{\beta'} - \frac{1}{\alpha};$$

$$\therefore \; \frac{1}{x'} - \frac{1}{x''} = \frac{1}{\beta} - \frac{1}{\beta'}, \quad \text{or} \quad \frac{2}{SV} = \frac{1}{\beta} - \frac{1}{\beta'} = \frac{1}{SR'} - \frac{1}{Sr'};$$

therefore the axis of y is harmonically divided.

Similarly, the axis of x is harmonically divided.

If SR meets QP in U, then UP, UQ, UV are proportional to y', y'' and SV;

$$\therefore \; \frac{2}{UV} = \frac{1}{UP} + \frac{1}{UQ};$$

therefore PQ is harmonically divided.

9. Let PP' (fig. 141) be any chord; DD' the chord of the conjugate arc; x, y, x', y' the co-ordinates of P, P' respectively when the ellipse is referred to its principal diameters as axes; then the equation to PP' is

$$Y - y = \frac{y' - y}{x' - x}(X - x) = m(X - x);$$

and the co-ordinates of D, D' are

$$\frac{-ay}{b}, \quad \frac{bx}{a}; \quad \frac{-ay'}{b}, \quad \frac{bx'}{a};$$

therefore the equation to DD' is

$$Y - \frac{bx}{a} = \frac{\frac{bx'}{a} - \frac{bx}{a}}{-\left(\frac{ay'}{b} - \frac{ay}{b}\right)}\left(X + \frac{ay}{b}\right),$$

$$\text{or} \quad Y - \frac{bx}{a} = -\frac{b^2}{a^2 m}\left(X + \frac{ay}{b}\right);$$

$$\therefore \; mY + \frac{b^2}{a^2}X = \frac{b}{a}(mx - y);$$

and if PP' passes through a point α, β,

$$\beta - y = m(\alpha - x); \quad \therefore \; mx - y = m\alpha - \beta,$$

$$\text{or} \quad mY + \frac{b^2}{a^2}X = \frac{b}{a}(m\alpha - \beta);$$

which is satisfied independently of m, by making

$$Y = \frac{b}{a}\alpha, \quad X = -\frac{a}{b}\beta;$$

therefore DD' always passes through a fixed point whose co-ordinates are $\frac{-a}{b}\beta$, $\frac{b}{a}\alpha$.

10. Let a, a', a'' be the semi-diameters parallel to PR, RT, (fig. 142) TQ respectively; then

$$\frac{RP^2}{RS^2} = \frac{a^2}{a'^2}, \quad \frac{TQ^2}{TS^2} = \frac{a''^2}{a'^2};$$

$$\therefore \frac{RP}{RS} : \frac{TQ}{TS} :: a : a'',$$

and is constant for all positions of RT.

Also, if H be one of the foci, and RH, SH, TH be joined, we have

$$\angle RHS = \tfrac{1}{2} \angle PHS, \quad \angle SHT = \tfrac{1}{2} \angle SHQ;$$

therefore by addition $\angle RHT = \frac{1}{2} \angle PHQ$, and is constant.

11. Transform the equation to polar co-ordinates by making

$$x = \rho \frac{\sin (45 - \theta)}{\sin 45}, \quad y = \rho \frac{\sin \theta}{\sin 45};$$

$$\therefore \rho^2 \{(\cos\theta - \sin\theta)^2 + \sqrt{2} (\cos\theta - \sin\theta) \sin\theta + 2 \sin^2\theta\} = 1;$$

$$\text{or } \rho^2 \left\{1 - \sin 2\theta + \frac{\sin 2\theta - (1 - \cos 2\theta)}{\sqrt{2}} + 1 - \cos 2\theta\right\} = 1;$$

$$\therefore \rho^2 \left\{\frac{2\sqrt{2} - 1}{\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}}\right) \sin 2\theta - \left(1 - \frac{1}{\sqrt{2}}\right) \cos 2\theta\right\} = 1;$$

$$\text{hence } \rho^2 \left\{\frac{2\sqrt{2} - 1}{\sqrt{2} - 1} - (\sin 2\theta + \cos 2\theta)\right\} = 2 + \sqrt{2};$$

$$\therefore \rho^2 \{3 + \sqrt{2} - \sqrt{2} \cos (2\theta - 45)\} = 2 + \sqrt{2};$$

$$\text{or } \rho^2 \left[1 + \frac{3}{\sqrt{2}} - \left\{2 \cos^2 \left(\theta - \frac{\pi}{8}\right) - 1\right\}\right] = \sqrt{2} + 1;$$

$$\text{hence } \rho^2 \left\{ (3 + 2\sqrt{2}) - 2\sqrt{2}\cos^2\left(\theta - \frac{\pi}{8}\right) \right\} = 2 + \sqrt{2};$$

$$\therefore\ \rho^2 \left\{ 1 - 2\sqrt{2}\,(3 - 2\sqrt{2})\cos^2\left(\theta - \frac{\pi}{8}\right) \right\} = (2 - \sqrt{2}),$$

which is the equation to an ellipse whose eccentricity

$$= \sqrt{2\sqrt{2}\,(3 - 2\sqrt{2})} = \sqrt[4]{2}\,(2 - \sqrt{2});$$

semi-axis-minor $= \sqrt{2 - \sqrt{2}}$;

and semi-axis-major $= \sqrt{\dfrac{2 - \sqrt{2}}{1 - e^2}} = \sqrt{\dfrac{2 - \sqrt{2}}{9 - 6\sqrt{2}}}$

$$= \sqrt{\tfrac{1}{3}(2 - \sqrt{2})(3 + 2\sqrt{2})} = \sqrt{\frac{2 + \sqrt{2}}{3}}.$$

The axis-major is inclined at an angle $\frac{\pi}{8}$ to the axis of x, and bisects the angle between the co-ordinate axes.

ST JOHN'S COLLEGE. Dec. 1845. (No. XVI.)

1. If two triangles have two sides of the one equal to two sides of the other each to each; but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other.

2. Define similar rectilineal figures, and shew that parallelograms about the diameter of any parallelogram are similar to the whole and to one another.

3. Explain briefly the advantages gained by the application of analysis to the solution of geometrical problems. If a line of given length in a given direction is represented by a, shew how a line of the same length, inclined at an angle θ to the former may properly be represented, and apply the method to express the cosine of an angle of a triangle in terms of the sides.

4. Through any point of a chord of a circle other chords are drawn; shew that lines from the middle point of the first chord to the middle points of the others, will meet them all at the same angle.

5. A tangent at any point P of an ellipse meets the major-axis produced in T, and perpendiculars upon it from the centre and focus in Y, Z; shew that TY has to PY the duplicate ratio of TZ to PZ.

6. The diameters of circles described about and within a semi-ellipse bounded by the minor axis are D, d; shew that D is a third proportional to AC and AB, and d a fourth proportional to AC, BC, and SH.

7. An ellipse and a pair of conjugate hyperbolas are described upon the same principal axes, and at the points

where any line through the centre meets the ellipse and one of the hyperbolas tangents are applied; find the locus of their intersection, and determine all its points of contact with the ellipse.

8. ACP, $A'CP'$ are two diameters of a circle; tangents at P, P' meet the diameters produced in B, B', and two parabolas are described touching them in A and B, A' and B' respectively. Shew that the quadrilateral formed by the two diameters and the axes of the parabolas may be inscribed in a circle. Find also the smallest possible value of the diameter of this circle, and the condition that it shall equal that of the original circle.

9. Trace the curve whose equation is

$$3x^2 - 10xy - 8y^2 + 4x - 2y + 1 = 0.$$

10. Two cones having their vertices coincident, their axes at right angles, and their surfaces in contact, are cut by a plane parallel to both axes: find the condition that the section shall be a pair of conjugate hyperbolas.

11. A straight line is drawn through a fixed point O, meeting a curve in the points $P_1 P_2 \ldots$ &c.; and in this line a point Q is taken, such that $OQ^{-n} = \Sigma OP^{-n}$. Determine the nature of the original curve, that, when n is any positive number, the locus of Q may be a similar curve.

12. The double tangents of the curve whose equation is

$$x^4\left(\frac{1}{a^4}+\frac{1}{c^4}\right)+y^4\left(\frac{1}{b^4}+\frac{1}{c^4}\right)+2x^2y^2\left(\frac{1}{c^4}-\frac{1}{a^2b^2}\right)$$
$$-2x^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)-2y^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+2=0,$$

touch it in points, all of which lie on a circle whose centre is the origin of co-ordinates.

13. Parabolas are described touching two lines at right angles to each other; shew that if the chords through the

points of contact are parallel to one another, the locus of the vertices of the parabolas is a straight line.

14. $P = 0$, $Q = 0$, $R = 0$, are the equations to three curves, all of which pass through the same point. Shew that of the curves whose equations are

$$P = a . Q . R, \quad P^2 = b . Q . R,$$

the former will in general touch the curve whose equation is $P = 0$, at that point, but the latter will not, a and b being constants. Exemplify this when

$$P = y - x, \quad Q = y - mx, \quad R = y + mx.$$

Why must the words *in general* be used?

15. With the asymptotes of a hyperbola as conjugate diameters ellipses are described touching the hyperbola. From any point of the hyperbola a pair of tangents is drawn to one of the ellipses; shew that tangents applied at the two points where the chord through the points of contact meets any other of the ellipses, will intersect in the same hyperbola.

SOLUTIONS TO (No. XVI.)

1. Euclid, Prop. 24, Book I.

2. Euclid, Def. 1, Book VI. and Prop. 24, Book VI.

3. See Peacock's Algebra, Vol. II. Arts. 823—831.

4. Let A (fig. 143) be a point in the first chord whose middle point is B: D the middle point of any other chord passing through A; C the centre of the circle; join CD, CB: then angles CDA, CBA are right angles, because D, B are the middle points of the chords; therefore a circle described on the diameter AC will pass through B and D; and

$$\angle ADB = \angle ACB$$

is constant for all positions of AD.

5. Let C be the centre, S the focus (fig. 144); draw the normal PK meeting CST in K: then

$$TY : YP :: TC : CK :: \frac{a^2}{x} : e^2x :: a^2 : e^2x^2;$$

and $$TZ : PZ :: TS : SK :: \frac{a^2}{x} - ae : ae - e^2x :: a : ex;$$

$$\therefore TY : YP :: TZ^2 : PZ^2.$$

6. Let O (fig. 145) be the centre of the circle described about the semi-ellipse, then $OB = OA$: let $OC = x$;

$$\therefore \sqrt{b^2 + x^2} = (a - x), \text{ or } b^2 = a^2 - 2ax;$$

$$\therefore x = \frac{a^2 - b^2}{2a}, \text{ and } AO = a - x = \frac{a^2 + b^2}{2a},$$

$$\text{or } D = \frac{a^2 + b^2}{a} = \frac{AB^2}{AC}.$$

Again, let o be the centre of the circle inscribed in the semi-ellipse; draw the normal oP and PM perpendicular to AC, then $oP = oC$: let $CM = x'$, $MP = y'$;

$$\therefore\ oM = \frac{b^2}{a^2}x' = (1 - e^2)\,x', \text{ and } Co = e^2 x';$$

hence $e^4 x'^2 = (1 - e^2)^2 x'^2 + y'^2 = (1 - e^2)^2 x'^2 + (1 - e^2)(a^2 - x'^2)$

$$= b^2 - e^2 x'^2 + e^4 x'^2;$$

$$\therefore\ x' = \frac{b}{e}, \text{ and } Co = e^2 x' = be;$$

$$\text{or } d = 2be = \frac{b\,.\,(2ae)}{a} = \frac{BC\,.\,SH}{AC}.$$

7. Let $y = mx$ be the equation to any line CPQ (fig. 146) drawn through the centre meeting the ellipse in P and the hyperbola in Q; then the equation to the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

and the equation to the tangent at P is $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} = 1$,

$$\text{but } x^2\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) = 1;$$

$$\therefore\ x' + \frac{a^2 m}{b^2}\,y' = \frac{a^2}{x} = \frac{a}{b}\sqrt{b^2 + a^2 m^2}: \quad (1)$$

also the equation to the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1,$$

and the equation to the tangent at Q is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = \pm 1, \text{ where } x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) = \pm 1;$$

$$\therefore\ x' - \frac{a^2 m}{b^2} y' = \pm \frac{a^2}{x} = \frac{a}{b}\sqrt{b^2 - a^2 m^2}, \text{ or } -\frac{a}{b}\sqrt{a^2 m^2 - b^2}, \quad (2)$$

according as Q is in the hyperbola or its conjugate.

Eliminate (m) between equations (1) and (2) and using the positive sign in (2) we have

$$\frac{b^2}{a^2}\left(x'^2 + \frac{2a^2 m}{b^2} x'y' + \frac{a^4 m^2}{b^4} y'^2\right) = b^2 + a^2 m^2;$$

$$\frac{b^2}{a^2}\left(x'^2 - \frac{2a^2 m}{b^2} x'y' + \frac{a^4 m^2}{b^4} y'^2\right) = b^2 - a^2 m^2;$$

therefore by addition,

$$\frac{2b^2}{a^2}\left(x'^2 + \frac{a^4 m^2}{b^4} y'^2\right) = 2b^2, \text{ or } x'^2 + \frac{a^4 m^2}{b^4} y'^2 = a^2;$$

by subtraction,

$$4m x' y' = 2a^2 m^2, \text{ or } a^2 m = 2x'y';$$

$$\therefore\ x'^2\left(1 + \frac{4y'^4}{b^4}\right) = a^2. \qquad (3)$$

Again, using the negative sign in equation (2)

$$\frac{b^2}{a^2}\left(x'^2 + \frac{2a^2 m}{b^2} x'y' + \frac{a^4 m^2}{b^4} y^2\right) = b^2 + a^2 m^2;$$

$$\frac{b^2}{a^2}\left(x'^2 - \frac{2a^2 m}{b^2} x'y' + \frac{a^4 m^2}{b^4} y^2\right) = a^2 m^2 - b^2;$$

therefore by addition,

$$\frac{2b^2}{a^2}\left(x'^2 + \frac{a^4 m^2}{b^4} y'^2\right) = 2a^2 m^2, \text{ or } x'^2 + \frac{a^4 m^2}{b^4} y'^2 = \frac{a^4 m^2}{b^2};$$

by subtraction, $4m x' y' = 2b^2$;

$$\therefore\ x'^2 + \frac{a^4}{4x'^2} = \frac{a^4 b^2}{4x'^2 y'^2}, \text{ or } y'^2\left(1 + \frac{4x'^4}{a^4}\right) = b^2; \qquad (4)$$

equations (3) and (4) are the loci of the intersections of the tangents at P and Q when CPQ meets the hyperbola and its conjugate respectively.

If $\dfrac{x'^2}{a'^2}\left(1+\dfrac{4y'^4}{b^4}\right)=1$; when the curve meets the ellipse

$$\left(1-\frac{y'^2}{b^2}\right)\left(1+\frac{4y'^4}{b^4}\right)=1, \text{ or } \frac{y'^2}{b^2}\left(1-\frac{2y'^2}{b^2}\right)^2=0; \quad (5)$$

$$\therefore\ y'=0, \text{ or } y'=\pm\frac{b}{\sqrt{2}},$$

and the corresponding values of x' are

$$x'=\pm a, \text{ and } x'=\pm\frac{a}{\sqrt{2}};$$

similarly, the curve $\dfrac{y'^2}{b'^2}\left(1+\dfrac{4x'^4}{a^4}\right)=1$ meets the ellipse when

$$x'=0, \text{ and } x'=\pm\frac{a}{\sqrt{2}};$$

and since equation (5) is a complete square, two values of y' become equal at the points determined by the equation; or the curves touch the ellipse at the six points in which

$$x'=0,\ x'=\pm a, \text{ and } x'=\pm\frac{a}{\sqrt{2}}.$$

8. Draw AQ, $A'Q$ perpendicular to AC, $A'C$ respectively (fig. 147); join $A'B'$, bisect it in D; also join CD; then CD will be parallel to the axis of the parabola which touches CA', CB': but $A'C=CP$, $A'D=DB'$; therefore CD is parallel to PB' and is therefore perpendicular to $A'C$: hence the tangent $A'C$ is perpendicular to the axis of the parabola; or A' is the vertex and $A'Q$ is the direction of the axis of the parabola.

Similarly, AQ is the direction of the axis of the parabola which touches AC, CB: and since the angles CAQ, $CA'Q$ are right angles, the circle described on the diameter CQ passes

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through the points A, A', and the quadrilateral $CAQA'$ may be inscribed in a circle.

Also, if $\angle PCP' = \alpha$, $CQ = AC \sec\frac{\alpha}{2}$ the least value of which is AC: and if $CQ = 2AC$, $\sec\frac{\alpha}{2} = 2$,

$$\therefore \cos\frac{\alpha}{2} = \tfrac{1}{2}, \text{ and } \alpha = 120^0.$$

9. $$x^2 - \left(\frac{10y-4}{3}\right)x = \frac{8y^2+2y-1}{3};$$

$$\therefore x^2 - \left(\frac{10y-4}{3}\right)x + \left(\frac{5y-2}{3}\right)^2 = \frac{49y^2-14y+1}{9};$$

$$\text{or } x - \frac{5y-2}{3} = \pm\left(\frac{7y-1}{3}\right);$$

$$\therefore 3x = 5y - 2 \pm (7y-1) = 12y - 3, \text{ or } -2y-1;$$

therefore the equation is reduced to two straight lines whose equations are

$$4y - x = 1, \text{ and } 2y + 3x + 1 = 0.$$

10. Let the two axes of the cones which are at right angles be taken for the axes of x and y; and a line through the vertex A perpendicular to them for the axis of z: then if α be the semi-vertical angle of the cone whose axis is Ax, $\frac{\pi}{2} - \alpha$ will be the semi-vertical angle of the cone whose axis is Ay: and if x, y, z be co-ordinates of any point in the first surface, x', y', z' co-ordinates of any point in the second surface:

$$y^2 + z^2 = x^2\tan^2\alpha, \text{ or } \frac{x^2}{z^2\cot^2\alpha} - \frac{y^2}{z^2} = 1; \quad (1)$$

$$x'^2 + z'^2 = y'^2\cot^2\alpha, \text{ or } \frac{x'^2}{z'^2\cot^2\alpha} - \frac{y'^2}{z'^2} = -\tan^2\alpha. \quad (2)$$

Now by making $z' = z$ constant, equations (1) and (2) are the equations of the sections to a cone made by a plane parallel to both axes, at a distance (z) from their plane: and if $a = \frac{\pi}{4}$, they manifestly become

$$\frac{x^2}{z^2} - \frac{y^2}{z^2} = 1,$$

$$\frac{x^2}{z^2} - \frac{y^2}{z^2} = -1;$$

the equations to two conjugate hyperbolas.

11. If the equation to the curve be referred to polar co-ordinates it will assume the form

$$\rho^m f_m(\theta) + \rho^{m-1} f_{m-1}(\theta) + \dots + \rho^2 f_2\theta + \rho f_1\theta + f_0 = 0.$$

When ρ is the radius vector, and $f_m(\theta)$, $f_{(m-1)}\theta \dots\dots f_1\theta$ homogeneous functions of $\sin\theta$ and $\cos\theta$ of m, $(m-1)$ &c. dimensions respectively, and f_0 a constant;

$$\therefore \frac{1}{\rho^m} + \frac{f_1(\theta)}{f_0}\frac{1}{\rho^{m-1}} + \frac{f_2(\theta)}{f_0}\frac{1}{\rho^{m-2}} + \dots + \frac{f_m(\theta)}{f_0} = 0;$$

$$\therefore \Sigma\rho^{-1} = -\frac{f_1(\theta)}{f_0}$$

is a homogeneous function of $\sin\theta$ and $\cos\theta$ of 1 dimension;

$$\Sigma\rho^{-2} = -\frac{f_1(\theta)}{f_0}\Sigma\rho^{-1} - \frac{2f_2\theta}{f_0}$$

is a homogeneous function of $\sin\theta$ and $\cos\theta$ of 2 dimensions;

$$\Sigma\rho^{-3} = -\frac{f_1\theta}{f_0}\Sigma\rho^{-2} - \frac{f_2\theta}{f_0}\Sigma\rho^{-1} - \frac{3f_3(\theta)}{f_0}$$

a homogeneous function of $\sin\theta$ and $\cos\theta$ of 3 dimensions.

Similarly, $\Sigma\rho^{-n} = F_n(\theta)$ a homogeneous function of $\sin\theta$ and $\cos\theta$ of n dimensions; therefore the equation to the locus of Q is

$$\rho'^{-n} = a^{-n}F_n(\theta) = a^{-n}(\cos\theta)^n F(\tan\theta),$$

where $F(\tan\theta)$ is a rational and integral function of $\tan\theta$ of n dimensions; and in order that this curve may be similar to the original curve, the original curve must have for its equation

$$\left(\frac{b}{\rho}\right)^n = \cos^n(\theta - a)\, F\tan(\theta - a).$$

12. The equation may be put under the form

$$\left(\frac{x^2+y^2}{c^2}\right)^2 + \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 - 2\left(\frac{x^2+y^2}{c^2}\right) - 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + 2 = 0,$$

$$\text{or } \left(\frac{x^2+y^2}{c^2} - 1\right)^2 + \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 - 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + 1 = 0.$$

$$\text{Now } \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 - 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + 1 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)^2 - \left(\frac{2y}{b}\right)^2$$

$$= \left\{\frac{x^2}{a^2} - \left(\frac{y}{b} + 1\right)^2\right\}\left\{\frac{x^2}{a^2} - \left(\frac{y}{b} - 1\right)^2\right\}$$

$$= \left(\frac{x}{a} - \frac{y}{b} - 1\right)\left(\frac{x}{a} - \frac{y}{b} + 1\right)\left(\frac{x}{a} + \frac{y}{b} + 1\right)\left(\frac{x}{a} + \frac{y}{b} - 1\right);$$

$$\therefore \left(\frac{x^2+y^2}{c^2} - 1\right)^2 + \left(\frac{x}{a} + \frac{y}{b} + 1\right)\left(\frac{x}{a} + \frac{y}{b} - 1\right)\left(\frac{x}{a} - \frac{y}{b} + 1\right)\left(\frac{x}{a} - \frac{y}{b} - 1\right) = 0;$$

therefore the four straight lines

$$\frac{x}{a} + \frac{y}{b} + 1 = 0, \quad \frac{x}{a} + \frac{y}{b} - 1 = 0,$$

$$\frac{x}{a} - \frac{y}{b} + 1 = 0, \quad \frac{x}{a} - \frac{y}{b} - 1 = 0$$

are double tangents, and meet the curve in the circle whose equation is $x^2 + y^2 = c^2$.

13. The line joining the points of contact will always pass through the focus of the parabola; suppose PSp (fig. 148) a chord passing through the focus; A the vertex; PQ, Qp tangents at P, p meeting in Q; produce PQ to meet the axis AS in T; draw PM perpendicular to AS, and let

$$\angle PTA = \theta, \quad AM = x, \quad MP = y;$$

$$\text{then } \tan\theta = \frac{y}{2x} = \sqrt{\frac{m}{x}};$$

$$\therefore\ x = m\cot^2\theta, \quad y = 2m\cot\theta, \quad \angle SPQ = \angle STP = \theta;$$

and the perpendicular from A on PQ

$$= \alpha = AT\sin\theta = x\sin\theta = m\frac{\cos^2\theta}{\sin\theta}.$$

Similarly, the perpendicular from A on Qp

$$= \beta = m\frac{\sin^2\theta}{\cos\theta};$$

$$\text{hence } \frac{\beta}{\alpha} = \tan^3\theta = \cot^3 QpP.$$

Now α, β are the co-ordinates of A referred to the fixed axes Qp, QP; and if $\angle QpP$ be constant, the locus of A is a straight line whose equation is $\beta = \cot^3 QpP\,.\,\alpha$.

14. Let $u = y - \beta - m(x - \alpha) = 0$ be the equation to a straight line passing through a point α, β; and $P = 0$ the equation to a curve passing through the same point; then if $\beta + m(x - \alpha)$ be substituted for y in $P = 0$, the resulting equation will be of the form $(x - \alpha) P' = 0$, since one value of x is α; P' will be a function of x and m; and by putting $x = \alpha$ in the equation $P' = 0$, an equation will be determined in terms of m; and if the values of m found from this equation make P' a multiple of $(x - \alpha)$, $P = 0$ becomes of the form $(x - \alpha)^2 P'' = 0$, and two values of x will in this case coincide, or $u = 0$ becomes a tangent to the equation $P = 0$.

If the equation $P = 0$ should be of such a form as to make the resulting equation after the substitution of $\beta + m(x - a)$ of the form $(x - a)^2 P' = 0$ for every value of m, which will always be the case if P be a rational function of $x - a$, and $y - \beta$ of which no term is of less than two dimensions, $u = 0$ will not be a tangent except for those values of m which make P' a multiple of $x - a$, since two values of x will only be made to coincide on this supposition, the equation $(x - a)^2 P' = 0$ indicating that two branches of the curve intersect when $x = a$; hence in this case in order that $u = 0$ may be a tangent to $P = 0$ we must have $P = (x - a)^3 P'' = 0$.

Hence when $u = 0$ is a tangent to $P = 0$, if $u = 0$ be substituted in $P - aQR = 0$, the resulting equation will be

$$(x - a)^2 P' - a(x - a) Q' . (x - a) R' = 0;$$

$$\text{or } (x - a)^2 (P' - a Q'R') = 0;$$

or $u = 0$ will in general be a tangent to $P - aQR = 0$, as well as $P = 0$; or the curves will touch one another.

If $u = 0$ when substituted in $P - aQR = 0$ gives the result

$$(x - a)^2 (P' - a Q'R') = 0$$

for every value of m; the value of m which makes $u = 0$ a tangent to $P = 0$, will not make $u = 0$ a tangent to $P - aQR = 0$, unless it makes $P' - aQ'R'$ at the same time a multiple of $x - a$, or the two curves will not touch one another unless the equation $u = 0$ to the tangent to $P = 0$ when combined with $P - aQR = 0$ gives a result $(x - a)^3 R'' = 0$.

If $u = 0$ be substituted in $P^2 - bQR = 0$, every value of m will give the result

$$(x - a)^2 (P'^2 - b Q'R') = 0;$$

and the value of m, which makes $u = 0$ a tangent to $P = 0$, will not make $u = 0$ a tangent to $P^2 - bQR = 0$, unless $P'^2 - bQ'R'$ be a multiple of $x - a$; or unless $u = 0$ when combined with $P^2 - bQR = 0$ gives the result $(x - a)^3 R'' = 0$. Hence $P = 0$ does not touch the curve $P^2 - bQR = 0$.

(β) If $y^2 - m^2x^2 - a(y - x) = 0$; when $y - x = 0$, we have $(1 - m^2)x^2 = 0$; or two values of x become $= 0$; and $y - x = 0$ touches the curve $y^2 - mx^2 - a(y - x) = 0$.

If $y^2 - m^2x^2 - b(y - x)^2 = 0$, this represents the equation to two straight lines passing through the origin; for

$$\left(\frac{y}{x}\right)^2 - m^2 - b\left(\frac{y}{x} - 1\right)^2 = 0,$$

$$\text{or } (1 - b)\left(\frac{y}{x}\right)^2 - b\left(\frac{y}{x}\right) + 1 - m^2 = 0;$$

therefore $\frac{y}{x}$ has two constant values; and neither of the lines will touch or coincide with $y - x = 0$ unless the value of one of the constants is unity.

15. Let a', b' be the semi-conjugate diameters of one of the ellipses drawn to touch the hyperbola in a point P (fig. 149) whose co-ordinates referred to the asymptotes are x', y'; TPt the common tangent, C the centre; then because TPt is a tangent to the ellipse

$$CT = \frac{a'^2}{x'}; \quad Ct = \frac{b'^2}{y'};$$

also because TPt is a tangent to the hyperbola,

$$CT = 2x', \quad Ct = 2y'; \quad \therefore \frac{a'^2}{x'} = 2x'; \quad \frac{b'^2}{y'} = 2y',$$

$$\text{or } a'^2 = 2x'^2; \quad b'^2 = 2y'^2,$$

$$\text{and } a'^2b'^2 = 4x'^2y'^2 = 4m^4; \quad \therefore a'b' = 2m^2;$$

similarly, if a'', b'' be the semi-conjugate diameters of any other ellipse, $a''b'' = 2m^2$.

Let tangents be drawn from a point h, k in the hyperbola to touch the ellipse whose equation is

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1;$$

then the equation to the straight line joining the points of contact is

$$\frac{hx}{a'^2} + \frac{ky}{b'^2} = 1; \quad (1)$$

and if tangents be drawn from a point x_1, y_1 to touch an ellipse whose equation is

$$\frac{x^2}{a''^2} + \frac{y^2}{b''^2} = 1,$$

the equation to the straight line joining the points of contact is

$$\frac{x_1 x}{a''^2} + \frac{y_1 y}{b''^2} = 1; \quad (2)$$

and in order that equations (1) and (2) may be the equations to the same straight line,

$$\frac{h}{a'^2} = \frac{x_1}{a''^2}; \quad \frac{k}{b'^2} = \frac{y_1}{b''^2}; \quad \therefore \frac{x_1 y_1}{a''^2 b''^2} = \frac{hk}{a'^2 b'^2} \text{ or } \frac{x_1 y_1}{2m^2} = \frac{m^2}{2m^2};$$

therefore $x_1 y_1 = m^2$; which shews that x_1, y_1 are co-ordinates of a point in the hyperbola.

ST JOHN'S COLLEGE. Dec. 1846. (No. XVII.)

1. Magnitudes have the same ratio to one another which their equimultiples have.

Give Euclid's definition of equal ratios. Explain why the properties proved in Book v. by means of *lines*, are true of *any* concrete magnitudes.

2. If two straight lines be at right angles to the same plane, they are parallel.

3. With the four lines which contain $a + b$, $a + c$, $a - b$, $a - c$ units respectively, construct a quadrilateral capable of having a circle inscribed in it.

Prove that no parallelogram can be inscribed in a circle except a rectangle; and that no parallelogram can be described about a circle except a rhomb.

4. From two fixed points draw lines to the same point of a fixed line, such that the tangents of the angles which they make with the fixed line are as the perpendicular distances of the points from it. Also when the tangents are in any other ratio.

5. In the circle, of which AB is the diameter, take any point P; and draw PC, PD on opposite sides of AP and equally inclined to it, meeting AB in C, D. Prove that

$$AC : BC :: AD : BD.$$

6. Two similar ellipses have a common vertex and a common direction of major-axes: a common tangent meets them in P, Q; and a perpendicular to their major-axes through the vertex meets PQ in O. Prove that $OP = OQ$.

7. The circle described from an extremity of the minor-axis of an ellipse, with radius equal to the distance of either directrix from the centre, will touch the ellipse in two points, one point, or not at all, as the eccentricity is greater, equal to, or less than $\frac{1}{2}\sqrt{2}$.

8. In two hyperbolas concentric and similarly situated, take two points whose abscissas are as the real axes of the hyperbolas. Prove that the locus of the middle points of the line joining them is a hyperbola concentric and similarly situated; and that the real axes, as also the imaginary axes, of the three hyperbolas, are in arithmetical progression.

9. The lengths (a, b) of two tangents to a parabola at right angles, are connected by the equation

$$\frac{a^{\frac{2}{3}}}{b^{\frac{4}{3}}} + \frac{b^{\frac{2}{3}}}{a^{\frac{4}{3}}} = \frac{1}{(\frac{1}{4}\text{ latus rectum})^{\frac{2}{3}}}.$$

10. If from the focus of a hyperbola as centre, a circle be described with a diameter equal to the imaginary axis, it will touch the asymptotes in the points where the nearer directrix meets them.

11. The quadrilateral $PQRS$ is inscribed in a circle. Join two opposite angles P, R; draw perpendiculars from S on PQ, PR, QR: the feet of these perpendiculars are in the same straight line.

12. Two ordinates of a parabola meet the axis at points equidistant from the focus. If the vertex be joined with the point where one of the ordinates meets the parabola; find the equation to the locus of the point where this line intersects the other ordinate; and trace the curve.

13. Two tangents are drawn to a parabola making angles θ, θ' with the axis. Prove that (1) if $\sin\theta \,.\, \sin\theta'$ be constant the locus of the intersections of the tangents is a circle whose centre is in the focus; (2) if $\tan\theta \,.\, \tan\theta'$ be constant the locus is a straight line perpendicular to the axis; (3) if $\cot\theta + \cot\theta'$ be constant the locus is a straight line parallel to the axis; (4) if $\cot\theta - \cot\theta'$ be constant the locus is a parabola equal to the original parabola.

14. Any three tangents to a parabola, the tangents of whose inclinations to the axes are in harmonical progression, will form a triangle of constant area.

15. In two ellipses (or hyperbolas) concentric and similarly situated, take two points P, Q whose abscissas are as their major-axes; and P', Q' two other such points. If $\theta\theta'\phi\phi'$ be the angles which the tangents at $PP'QQ'$ make with the axes; prove that $\frac{\tan\theta}{\tan\theta'} = \frac{\tan\phi}{\tan\phi'}$.

If the curves be likewise confocal, prove that $PQ' = P'Q$.

16. Any number of ellipses (or hyperbolas) concentric, similar, and similarly situated, are intersected by a line parallel to a directrix in $PP'P''$...: prove that the extremities of the diameters respectively conjugate to the diameters through $PP'P''$... are in a line perpendicular to the directrix.

17. If the above curves be cut by any concentric hyperbola, whose asymptotes have the same direction as their axes, in $QQ'Q''$...: prove that the extremities of the diameters respectively conjugate to the diameters through $QQ'Q''$... are situated in another branch of the hyperbola.

18. An ellipse being traced on a plane; the vertices of all the right cones of which it might be a section, are situated in a hyperbola whose imaginary axis is equal to the axis-minor of the ellipse, and real axis equal to the distance between its foci.

And conversely, the locus of the vertices of all the right cones, of which this hyperbola might be a section, is the original ellipse.

19. Find the volume of the pyramid of least volume which can be formed by three planes touching a given right cone, and the plane of the cone's base.

SOLUTIONS TO (No. XVII.)

1. Euclid, Prop. 15, Book v. Potts' Euclid, Def. 5, Book v. and note to the definition, p. 162.

2. Euclid, Prop. 9, Book xi.

3. (1) Let $ABCD$ (fig. 150) be a quadrilateral figure touching a circle in the points a, b, c, d; then

$$Aa = Ad,\ \ Ba = Bb,\ \ Cc = Cb,\ \ Dc = Dd;$$

$$\therefore\ Aa + Ba + Cc + Dc = Ad + Dd + Bb + Cb,$$

$$\text{or}\ \ AB + CD = BC + AD;$$

hence the sum of the two opposite sides is equal to the sum of the remaining two sides. Hence take AC any line less than $(a - b + a - c)$; describe a triangle ABC having $AB = a + b$, $BC = a + c$; and upon AC describe a triangle ADC, having $CD = a - b$, $DA = a - c$, then a circle may be inscribed in the quadrilateral figure $ABCD$.

(2) Let AB, CD (fig. 151) be two equal and parallel chords of a circle whose centre is E, then $ABCD$ will be a parallelogram; from E draw EF perpendicular to AB, and produce FE to meet CD in G; then $\angle EGD = \angle EFA =$ a right angle; and

$$FB = \frac{AB}{2} = \frac{CD}{2} = DG;$$

hence FG is parallel to BD, and the angles at B and D are right angles; similarly, the angles at A and C are right angles.

(3) If a quadrilateral be described about a circle, the sum of the opposite sides is equal to the sum of the remaining sides; and if the quadrilateral be a parallelogram whose sides are a, b; we have $2a = 2b$; $\therefore\ a = b$, or the parallelogram is a rhombus.

4. Let A, B (fig. 152) be the given points; draw AM, BN perpendicular to the given line; produce BN to P, and take $NP : BN :: n : 1$, where n is the ratio of the tangents; join AP, meeting MN in Q; and draw BQ; then

$$\frac{\tan \angle AQM}{\tan \angle BQN} = \frac{\tan \angle PQN}{\tan \angle BQN} = \frac{PN}{BN} = n.$$

$$\text{If } \frac{\tan \angle AQM}{\tan \angle BQN} = \frac{AM}{BN};$$

$$\text{then } \frac{AM}{BN} = \frac{PN}{BN}, \quad \text{or } PN = AM;$$

hence in this case make $PN = AM$.

5. Since angle CPD (fig. 153) is bisected by PA,

$$CA : AD :: CP : PD;$$

and since angle APB is a right angle, PB bisects the exterior angle of the triangle CPD;

$$\therefore CP : PD :: CB : BD;$$

hence $CA : AD :: CB : BD$, or $AC : BC :: AD : BD$.

6. Let A (fig. 154) be the common vertex, CD, $C'D'$ the semi-diameters of the two ellipses parallel to PQ; CB, $C'B'$ the semi-diameters parallel to AO; then since the ellipses are similar

$$\frac{CD}{CB} = \frac{C'D'}{C'B'}; \text{ but } \frac{PO}{AO} = \frac{CD}{CB} = \frac{C'D'}{C'B'} = \frac{QO}{AO}; \quad \therefore PO = QO.$$

7. The equations to the circle and ellipse are

$$x^2 + (y + b)^2 = \frac{a^2}{e^2}, \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

hence at the points in which the circle meets the ellipse,

$$a^2 - \frac{a^2}{b^2} y^2 + y^2 + 2by + b^2 = \frac{a^2}{e^2},$$

or $\left(\dfrac{1}{1-e^2}-1\right)y^2-2by=-a^2\left\{\dfrac{1-e^2}{e^2}-(1-e^2)\right\}=-b^2\left(\dfrac{1-e^2}{e^2}\right)$;

$$\therefore\ \frac{e^2y^2}{1-e^2}-2by+\frac{1-e^2}{e^2}b^2=0;$$

$$\therefore\ \frac{eay}{b}-\frac{b^2}{ea}=0,\ \text{ or }\ y=\frac{b^3}{e^2a^2};$$

and since y has only one value, the ellipse and circle will in general touch one another in the two points in which

$$y=\frac{b^3}{e^2a^2}:\ \text{ or }\ \left(\frac{x}{a}\right)^2=1-\left(\frac{y}{b}\right)^2=1-\frac{b^4}{e^4a^4};$$

$$\therefore\ x=\pm\frac{\sqrt{e^4a^4-b^4}}{e^2a}.$$

If $ea<b$, or $e<\sqrt{1-e^2}$, and therefore $e<\dfrac{\sqrt{2}}{2}$ the values of x and y are impossible, in which case the circle will not meet the ellipse. If $ea=b$, or $e=\dfrac{\sqrt{2}}{2}$; the two points coincide in the extremity of the axis-minor, and there is only one point of contact.

8. Let $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, $\dfrac{x'^2}{a'^2}-\dfrac{y'^2}{b'^2}=1$ be the equations to two hyperbolas, and let $\dfrac{x'}{a'}=\dfrac{x}{a}$; then $\dfrac{y'}{b'}=\dfrac{y}{b}$; and if X, Y be the co-ordinates of the middle point of the chord joining the points x, y, x', y'; we have

$$X=\frac{x+x'}{2},\quad Y=\frac{y+y'}{2};$$

$$\text{or }\ X=\left(1+\frac{a'}{a}\right)\frac{x}{2},\quad Y=\left(1+\frac{b'}{b}\right)\frac{y}{2};$$

$$\therefore\ x=\frac{2aX}{a+a'},\quad y=\frac{2bY}{b+b'},\quad\text{and}\ \left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2=1;$$

$$\therefore \left\{\frac{X}{\frac{1}{2}(a+a')}\right\}^2 - \left\{\frac{Y}{\frac{1}{2}(b+b')}\right\}^2 = 1,$$

which is the equation to a hyperbola whose axes are $a + a'$, $b + b'$; which are arithmetic means between $2a$, $2a'$ and $2b$, $2b'$ respectively.

9. Let PSp (fig. 148) be any focal chord making an angle θ with the axis; PQ, Qp two tangents intersecting at right angles; $Qp = a$, $QP = b$; latus rectum $= 4m$; angle $PSM = \theta$;

$$\therefore \angle SPQ = \frac{\theta}{2}, \quad \text{and } SP = \frac{m}{\sin^2\frac{\theta}{2}};$$

$$Sp = \frac{m}{\cos^2\frac{\theta}{2}}, \quad Pp = \frac{m}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}};$$

$$\text{hence} \quad a = Pp \sin\frac{\theta}{2} = \frac{m}{\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}},$$

$$b = Pp\cos\frac{\theta}{2} = \frac{m}{\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}};$$

$$\text{or} \quad \frac{a}{b^2} = \frac{\sin^3\frac{\theta}{2}}{m}, \quad \frac{b}{a^2} = \left(\frac{\cos^3\frac{\theta}{2}}{m}\right);$$

$$\therefore \frac{a^{\frac{2}{3}}}{b^{\frac{4}{3}}} + \frac{b^{\frac{2}{3}}}{a^{\frac{4}{3}}} = \frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{m^{\frac{2}{3}}} = \frac{1}{m^{\frac{2}{3}}}.$$

10. The equation to the circle is $(x - ae)^2 + y^2 = b^2$; and to the asymptotes $y = \pm\frac{b}{a}x$; hence at the points in which the circle meets the asymptotes

$$(x - ae)^2 + (e^2 - 1)x^2 = a^2(e^2 - 1);$$

$$\therefore e^2x^2 - 2aex + a^2 = 0, \text{ or } (ex - a)^2 = 0;$$

which shews that the circle touches the asymptotes in two points, the abscissa of which is $\frac{a}{e}$, and the ordinates $\frac{b}{e}$ and $-\frac{b}{e}$ respectively; or at the point in which the directrix meets them, since for every point in the directrix $x = \frac{a}{e}$.

11. If from any point in the circumference of a circle circumscribing a triangle perpendiculars be drawn upon the three sides, the feet of the perpendiculars will be in the same right line (Ex. 15. No. XIV); and S is a point in the circle circumscribing the triangle PQR, hence the feet of the three perpendiculars drawn from S are in the same straight line.

12. Let MP, $M'P'$ (fig. 155) be the two ordinates; draw AP' intersecting MP in Q; $AM = x$, $MQ = y$; $AM' = x'$, $M'P' = y'$; then $x + x' = 2a$;

$$\text{and } \frac{y}{x} = \frac{y'}{x'} = 2\sqrt{\frac{a}{x'}} = 2\sqrt{\frac{a}{2a - x}}; \quad \therefore \frac{y^2}{x^2} = \frac{4a}{2a - x}$$

is the equation required.

When $x = 0$, $y = 0$ or the curve passes through the origin, and limit $\frac{y}{x} = \pm\sqrt{2}$; as x increases y increases, and when $x = 2a$, y is infinite, or the ordinate at a distance $2a$ from the origin is an asymptote. When $x > 2a$, y is impossible.

When x is negative, $\frac{y^2}{x^2} = \frac{4a}{2a + x}$; as x increases y increases, and when x is very large, the curve approaches to the parabola $y^2 = 4a(x - 2a)$ as the asymptotic curve. The curve is that traced in (fig. 156).

13. (1) Let $y^2 = 4ax$ be the equation to the parabola; then $yy' = 2a(x + x')$ is the equation to the tangent, and if

$$\frac{2a}{y} = \tan\theta = m; \; y' = m\left(x' + \frac{a}{m^2}\right);$$

or $m^2 - \frac{y'}{x'} m + \frac{a}{x'} = 0$; (1) but when $\sin\theta \sin\theta' = \alpha$;

$$\operatorname{cosec}^2\theta \operatorname{cosec}^2\theta' = \frac{1}{\alpha^2}, \text{ or } \left(\frac{1}{m^2} + 1\right)\left(\frac{1}{m'^2} + 1\right) = \frac{1}{\alpha^2};$$

$$\text{hence } \frac{1}{m^2} + \frac{1}{m'^2} + \frac{1}{m^2 m'^2} = \frac{1}{\alpha^2} - 1;$$

$$\therefore \left(\frac{1}{m} + \frac{1}{m'}\right)^2 + \left(\frac{1}{mm'} - 1\right)^2 = \frac{1}{\alpha^2};$$

$$\text{and } \frac{1}{m} + \frac{1}{m'} = \frac{y'}{a}; \quad \frac{1}{mm'} = \frac{x'}{a};$$

$$\therefore \left(\frac{y'}{a}\right)^2 + \left(\frac{x' - a}{a}\right)^2 = \frac{1}{\alpha^2};$$

$$\text{hence } (x' - a)^2 + y'^2 = \frac{a^2}{\alpha^2}$$

which is the equation to a circle whose centre is the focus, and radius $= \frac{a}{\alpha} = a \operatorname{cosec}\theta \operatorname{cosec}\theta'$.

(2) Let $\tan\theta \tan\theta' = \beta$; $\therefore mm' = \frac{a}{x'} = \beta$; hence $x' = \frac{a}{\beta}$ which is the equation to a straight line perpendicular to the axis at a distance $a \cot\theta \cot\theta'$ from the origin.

(3) Let $\cot\theta + \cot\theta' = \gamma$; $\therefore \frac{1}{m} + \frac{1}{m'} = \gamma$; or $\frac{y'}{a} = \gamma$; which is the equation to a straight line parallel to the axis at a distance $a\gamma$ from the axis.

(4) From equation (1)

$$\frac{1}{m} = \frac{y'}{2a} + \frac{\sqrt{y'^2 - 4ax'}}{2a};$$

$$\frac{1}{m'} = \frac{y'}{2a} - \frac{\sqrt{y'^2 - 4ax'}}{2a};$$

$$\therefore \frac{1}{m} - \frac{1}{m'} = \cot\theta - \cot\theta' = \frac{\sqrt{y'^2 - 4ax'}}{a} = \delta;$$

$$\therefore y'^2 - 4ax' = a^2\delta^2; \text{ hence } y'^2 = 4a\left(x' - \frac{a\delta^2}{4}\right);$$

which is the equation to a parabola whose latus rectum $= 4a$, and vertex is at a distance $\frac{a\delta^2}{4}$ from the origin.

14. Let P, Q, R (fig. 157) be three points of a parabola whose ordinates are y, y', y''; θ, θ', θ'' the inclinations of the tangents at P, Q, R to the axis of the parabola; then since the tangents are in harmonic progression,

$$\cot\theta + \cot\theta'' = 2\cot\theta'; \quad \text{or} \quad \frac{y}{2a} + \frac{y''}{2a} = \frac{y'}{a};$$

hence $y + y'' = 2y'$, which shews that Q is the vertex of the diameter whose ordinate is PR.

Let the tangents at P and R intersect in T; then the tangent LQM is parallel to PR; $TQ = \frac{1}{2}TV$; and the triangles LMT, RPT are similar;

$$\therefore \triangle LMT = \tfrac{1}{4}\triangle RPT = \tfrac{1}{2}\triangle PQR = \tfrac{1}{2} QV.VR \sin QVR;$$

$$\text{but } \frac{4a}{\sin^2 QVR} QV = RV^2; \text{ and } RV.\sin QVR = \frac{y''-y}{2};$$

$$\therefore QV = \frac{1}{16a}(y''-y)^2;$$

$$\text{and } \triangle LMT = \frac{1}{16a}(y''-y)^2\frac{y''-y}{4} = \frac{(y''-y)^3}{64a}.$$

Hence $\triangle LMT$ is constant as long as $y''-y$ is constant; or for any three consecutive points in a series of points P, Q, R, S, &c. which have the tangents of the inclinations of the tangents to the axis of x in harmonic progression.

15. (1) Let x, y, x', y' be the co-ordinates of P, P', and X, Y, X', Y' the co-ordinates of Q, Q'; then

$$\tan\theta = -\frac{b^2 x}{a^2 y}; \quad \tan\theta' = -\frac{b^2 x'}{a^2 y'};$$

$$\therefore \frac{\tan\theta}{\tan\theta'} = \frac{x}{x'}\left(\frac{y'}{y}\right); \quad \text{similarly } \frac{\tan\phi}{\tan\phi'} = \frac{X}{X'}\cdot\frac{Y'}{Y};$$

$$\text{but } \frac{X}{a'} = \frac{x}{a}; \quad \therefore \frac{Y}{b'} = \frac{y}{b};$$

$$\text{also } \frac{X'}{a'} = \frac{x'}{a}; \quad \therefore \frac{Y'}{b'} = \frac{y'}{b};$$

$$\text{hence } \frac{x}{x'} = \frac{X}{X'}; \quad \frac{y}{y'} = \frac{Y}{Y'};$$

$$\text{or } \frac{\tan\theta}{\tan\theta'} = \frac{\tan\phi}{\tan\phi'}.$$

If the ellipses be concentric and confocal, they coincide; hence P, Q coincide, and P', Q' coincide; $\therefore PQ' = QP'$.

16. If x, y be the co-ordinate of one of the points as P; then the co-ordinates of D are $y' = \frac{b}{a}x$, and $\frac{b}{a}$ is constant for all similar ellipses; therefore y' is constant for all similar ellipses; and the locus of D is a line parallel to the axis, or perpendicular to the directrix.

17. If x, y be the co-ordinates of Q, so that $xy = m^2$; X, Y the co-ordinates of D; then

$$X = \pm\frac{a}{b}y; \quad Y = \frac{b}{a}x;$$

the positive or negative sign being used according as the series of curves are ellipses or hyperbolas;

$$\therefore XY = \pm xy = \pm m^2;$$

or the locus of D is the hyperbola, or the conjugate hyperbola, according as the series of curves are ellipses or hyperbolas.

18. The vertex of the cone will lie in a plane passing through the axis of the elliptic section, and perpendicular to its plane; hence the vertices of all the cones will lie in the same plane. Let S, H (fig. 158) be the extremities of the axis-major of the ellipse; PQ the axis of one of the cones; draw SY, HZ perpendicular to PQ; then if $2b$ be the axis-minor of the elliptic section, $SY \,.\, HZ = b^2$; and S and H are on different sides of PQ, therefore PQ always touches a hyperbola whose foci are S and H, and conjugate axis (b), and P is a point in this hyperbola, therefore the locus of P is a hyperbola whose foci are S, H. If a', b be the semi-axes of the hyperbola; S', H' the foci of the ellipse, then

$$SH^2 = a'^2 + b^2 = a^2\,; \ \therefore \ a'^2 = a^2 - b^2 = CS'^2 = CH'^2\,;$$

$$\therefore \ a' = CS' = CH'\,;$$

or the extremities of the axis of the hyperbola are H', S'.

Again, if $S'\,H'$ be the axis-major of a hyperbola, whose foci are S, H; then the vertex of the cone will be in a plane perpendicular to the plane of the hyperbola, and therefore in the plane of the ellipse whose axis-major is SH; and if $S'Y'$, $H'Z'$ be drawn perpendicular to the axis of the cone, $S'Y' \,.\, H'Z' = b^2$; or the locus of the vertex is an ellipse whose semi-axis-minor is b, and foci S', H', or it will be the original ellipse, since in this case $S'Y'$, $H'Z'$ are on the same side of the axis of the cone.

19. The volume of the pyramid $= \frac{1}{3}$ (altitude of cone $\times$ triangular base); and the area of the triangle which touches the circular base $= r\dfrac{(a+b+c)}{2}$, where a, b, c are the sides of the triangle, and r the radius of the base of the cone; this will be least when $a + b + c$ is least; but

$$\frac{a+b+c}{2} = r\left\{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right\};$$

and if A be given

$$\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{2\sin\dfrac{B+C}{2}}{\cos\dfrac{B-C}{2}-\cos\dfrac{B+C}{2}}=\frac{2\cos\dfrac{A}{2}}{\cos\dfrac{B-C}{2}-\sin\dfrac{A}{2}},$$

which will be least when $\cos\dfrac{B-C}{2}$ is greatest, or when $B=C$.

Hence the triangle will be least possible when $A=B=C$.

In this case the area of the triangular base

$$=3r^2(\tan 60)=3\sqrt{3}\,.\,r^2;$$

hence the volume of the pyramid

$$=\tfrac{1}{3}\left\{3\sqrt{3}r^2\right\}h=\sqrt{3}r^2h.$$

where h is the altitude of the cone.

APPENDIX I.

1. To inscribe a triangle in an ellipse, whose sides shall pass through three given points.

Let ABC (fig. 159) be the required triangle whose sides AB, AC, BC are to pass through the three points c, b, a, whose co-ordinates are a_3, b_3; a_2, b_2; a_1, b_1 respectively; let the centre O be assumed for the origin, and let x_1, y_1; x_2, y_2; x_3, y_3 be the co-ordinates of A, B, C, and $x_1 = a\cos\theta_1$; therefore $y_1 = b\sin\theta_1$; similarly, let $x_2 = a\cos\theta_2$, $x_3 = a\cos\theta_3$; therefore $y_2 = b\sin\theta_2$, $y_3 = b\sin\theta_3$; and the equation to AB is

$$y - b\sin\theta_1 = \frac{b\,(\sin\theta_2 - \sin\theta_1)}{a\,(\cos\theta_2 - \cos\theta_1)}(x - a\cos\theta_1);$$

$$\text{and if } \tan\frac{\theta_1}{2} = t_1;\ \tan\frac{\theta_2}{2} = t_2,\ \tan\frac{\theta_3}{2} = t_3,$$

$$y = -\frac{b}{a}\left(\frac{1 - t_1 t_2}{t_1 + t_2}\right)x + \frac{b\,(1 + t_1 t_2)}{t_1 + t_2};$$

and since AB passes through a point whose co-ordinates are a_3, b_3, we have

$$\frac{ab_3}{b}(t_1 + t_2) = -a_3(1 - t_1 t_2) + a(1 + t_1 t_2);$$

$$\text{or } (a + a_3)\,t_1 t_2 - \frac{a}{b}b_3(t_1 + t_2) + (a - a_3) = 0; \qquad (1)$$

similarly,

$$(a + a_2)\,t_1 t_3 - \frac{a}{b}b_2(t_1 + t_3) + a - a_2 = 0, \qquad (2)$$

$$(a + a_1)\,t_2 t_3 - \frac{a}{b}b_1(t_2 + t_3) + a - a_1 = 0. \qquad (3)$$

$$\text{Let } \frac{b}{a}\left(\frac{a + a_3}{b_3}\right) = m_3,\ \frac{b}{a}\left(\frac{a - a_3}{b_3}\right) = n_3,$$

$$\frac{b}{a}\left(\frac{a + a_2}{b_2}\right) = m_2,\ \frac{b}{a}\left(\frac{a - a_2}{b_2}\right) = n_2,$$

$$\frac{b}{a}\left(\frac{a+a_1}{b_1}\right)=m_1,\quad \frac{b}{a}\left(\frac{a-a_1}{b_1}\right)=n_1\,;$$

$$\text{hence } m_3t_1t_2-(t_1+t_2)+n_3=0,$$

$$m_2t_1t_3-(t_1+t_3)+n_2=0,$$

$$m_1t_2t_3-(t_2+t_3)+n_1=0\,;$$

$$\therefore\ t_3=\frac{t_1-n_2}{m_2t_1-1},\quad t_3=\frac{t_2-n_1}{m_1t_2-1},\ \text{ and } t_2=\frac{t_1-n_3}{m_3t_1-1},$$

$$\text{or }\ \frac{t_2-n_1}{m_1t_2-1}=\frac{(1-n_1m_3)\,t_1+n_1-n_3}{(m_1-m_3)t_1+1-m_1n_3}\,;$$

$$\therefore\ \frac{t_1-n_2}{m_2t_1-1}=\frac{(1-n_1m_3)\,t_1+(n_1-n_3)}{(m_1-m_3)\,t_1+(1-m_1n_3)}\,;$$

$$\text{hence }\ (m_1-m_2-m_3+n_1m_2m_3)\,t_1^2$$
$$+\{2-m_1(n_2+n_3)-n_1(m_2+m_3)+n_2m_3+m_2n_3\}t_1$$
$$+n_1-n_2-n_3+m_1n_2n_3=0\,;\qquad (4)$$

from which equation the two values of t_1 or $\tan\dfrac{\theta_1}{2}$ may be determined; which will give two positions of the point A, and by drawing AB through c, and AC through b, the position of the two triangles whose sides pass through the three given points will be determined.

Cor. When $b=a$ the ellipse becomes a circle; and a triangle will be inscribed in a circle having its three sides passing through three fixed points.

2. To find the equation to the straight line passing through the two positions of A.

Let the coefficients of equation (4) be A, B, C; then

$$A\tan^2\frac{\theta_1}{2}+B\tan\frac{\theta_1}{2}+C=0,$$

$$\text{or }\ A\sin^2\frac{\theta_1}{2}+B\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}+C\cos^2\frac{\theta_1}{2}=0\,;$$

$$\therefore\ A(1-\cos\theta_1)+B\sin\theta_1+C(1+\cos\theta_1)=0\,;$$

$$\text{hence } (C - A)\cos\theta_1 + B\sin\theta_1 + (A + C) = 0\,;$$

$$\therefore\ (C - A)\frac{x_1}{a} + B\,.\left(\frac{y_1}{b}\right) + (A + C) = 0\,;$$

or $\{(n_1 - m_1) - (n_2 - m_2) - (n_3 - m_3) + m_1 n_2 n_3 - n_1 m_2 m_3\}\dfrac{x_1}{a}$

$$+ \{2 - m_1(n_2 + n_3) - n_1(m_2 + m_3) + n_2 m_3 + m_2 n_3\}\frac{y_1}{b}$$

$$+ \{n_1 + m_1 - (n_2 + m_2) - (n_3 + m_3) + m_1 n_2 n_3 + n_1 m_2 m_3 = 0\,; \quad (5)$$

and the straight line represented by equation (5) will pass through the two positions of A, which will be determined by the intersection of the straight line with the ellipse.

3. Let a circle be described on the axis-major ED of the ellipse (fig. 160), and suppose a triangle ABC to be inscribed in this circle whose sides shall pass through the three points a', b', c' whose co-ordinates are

$$a_1,\ \frac{a}{b}b_1\,;\quad a_2,\ \frac{a}{b}b_2\,;\quad a_3,\ \frac{a}{b}b_3\,;$$

then using the same notation we shall obtain equations (1), (2), (3) for determining the values of t_1, t_2, t_3 in the circle; hence the values of θ_1, θ_2, θ_3 are the same as in the ellipse; and if AA', BB', CC' be drawn perpendicular to ED to meet the ellipse in A', B', C', the three sides of the triangle $A'B'C'$ will pass through the three points a, b, c whose co-ordinates are a_1, b_1; a_2, b_2; a_3, b_3 respectively.

4. To inscribe a triangle in a hyperbola whose sides shall pass through three fixed points.

Using the same notation as before, let

$$x_1 = a\sec\theta_1,\ \therefore\ y_1 = b\tan\theta_1\,;$$

similarly let

$$x_2 = a\sec\theta_2,\ x_3 = a\sec\theta_3\,;\ \therefore\ y_2 = b\tan\theta_2,\ y_3 = b\tan\theta_3\,;$$

and the equation to AB is

$$y - b\tan\theta_1 = \frac{b(\tan\theta_2 - \tan\theta_1)}{a(\sec\theta_2 - \sec\theta_1)}\{x - a\sec\theta_1\}\,;$$

$$\text{or } y = \frac{b}{a}\left(\frac{1 + t_1 t_2}{t_1 + t_2}\right) x - b\left(\frac{1 - t_1 t_2}{t_1 + t_2}\right);$$

$$\text{hence } \frac{a}{b} b_3 (t_1 + t_2) = (1 + t_1 t_2) a_3 - a (1 - t_1 t_2);$$

$$\therefore\ (a + a_3) t_1 t_2 - \frac{a}{b} b_3 (t_1 + t_2) + a_3 - a = 0,$$

$$\text{or} \quad m_3 t_1 t_2 - (t_1 + t_2) - n_3 = 0;$$

$$\text{similarly} \quad m_2 t_1 t_3 - (t_1 + t_3) - n_2 = 0;$$

$$m_1 t_2 t_3 - (t_2 + t_3) - n_1 = 0;$$

$$\text{hence } (m_1 - m_2 - m_3 - n_1 m_2 m_3) t_1^2$$
$$+ \{2 + m_1 (n_2 + n_3) + n_1 (m_2 + m_3) - n_2 m_3 - m_2 n_3\} t_1$$
$$- (n_1 - n_2 - n_3 - m_1 n_2 n_3) = 0;$$

from which the two values of t_1 may be determined.

5. To find the equation to the straight line passing through the two positions of A.

If $A t_1^2 + B t_1 + C = 0$; we have

$$(C - A) \cos\theta_1 + B \sin\theta_1 + (A + C) = 0,$$

$$\text{or } (A + C) \sec\theta_1 + B \tan\theta_1 + (C - A) = 0;$$

$$\therefore\ (A + C)\frac{x_1}{a} + B\frac{y_1}{b} + (C - A) = 0;$$

is the equation to the line joining the two positions of A.

6. To inscribe a triangle in a parabola whose sides shall pass through three fixed points.

Let ABC be the required triangle, whose sides AB, AC, BC are to pass through three points c, b, a whose co-ordinates measured from the vertex are a_3, b_3; a_2, b_2; a_1, b_1; let x_1, y_1; x_2, y_2; x_3, y_3; be the co-ordinates of A, B, and C respectively; then the equation to AB is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1);$$

or $y - \dfrac{y_2 - y_1}{x_2 - x_1} x = \dfrac{y_1 x_2 - x_1 y_2}{x_2 - x_1}$; but $y_1^2 = 4 a x_1$; $y_2^2 = 4 a x_2$;

$$\therefore\ y - \frac{4a}{y_1 + y_2}x = \frac{y_1 y_2}{y_1 + y_2};$$

and since this passes through the points a_3, b_3;

$$\therefore\ b_3 - \frac{4a}{y_1 + y_2}a_3 = \frac{y_1 y_2}{y_1 + y_2};$$

$$\text{or } y_1 y_2 - b_3(y_1 + y_2) + 4a a_3 = 0; \quad (1)$$

$$\text{similarly } y_1 y_3 - b_2(y_1 + y_3) + 4a a_2 = 0; \quad (2)$$

$$y_2 y_3 - b_1(y_2 + y_3) + 4a a_1 = 0; \quad (3)$$

$$\text{hence } y_3 = \frac{b_2 y_1 - 4a a_2}{y_1 - b_2}; \quad y_3 = \frac{b_1 y_2 - 4a a_1}{y_2 - b_1}; \quad y_2 = \frac{b_3 y_1 - 4a a_3}{y_1 - b_3};$$

$$\therefore\ \frac{b_1 y_2 - 4a a_1}{y_2 - b_1} = \frac{(b_1 b_3 - 4a a_1) y_1 + 4a(a_1 b_3 - a_3 b_1)}{(b_3 - b_1) y_1 + b_1 b_3 - 4a a_3} = \frac{b_2 y_1 - 4a a_2}{y_1 - b_2};$$

$$\text{or } \{b_1 b_3 - b_2 b_3 + b_1 b_2 - 4a a_1\} y_1^2$$
$$+ [4a\{(a_1 + a_2) b_3 - (a_2 + a_3) b_1 + (a_1 + a_3) b_2\} - 2 b_1 b_2 b_3] y_1$$
$$+ 4a\{a_2(b_1 b_3 - 4a a_3) - b_2(a_1 b_3 - a_3 b_1)\} = 0; \quad (4)$$

from which may be determined the two values of y_1 which will give the two positions of the point A.

7. To find the equation to the straight line passing through the two positions of A.

Since $y_1^2 = 4a x_1$; substituting this for y_1^2 in equation (4) we have

$$4a(b_1 b_3 - b_2 b_3 + b_1 b_2 - 4a a_1) x_1$$
$$+ [4a\{(a_1 + a_2) b_3 - (a_2 + a_3) b_1 + (a_1 + a_3) b_2\} - 2 b_1 b_2 b_3] y_1$$
$$+ 4a\{a_2(b_1 b_3 - 4a a_3) - b_2(a_1 b_3 - a_3 b_1)\} = 0,$$

which is the equation to the straight line passing through the two positions of A; and the intersection of this straight line with the parabola will give the two points required.

8. To construct the triangle geometrically when the three given points are in the same straight line.

Let m, n, p (fig. 35) be the three points; draw any line mab through m; through b drawn bcn, and through c draw pcd; join ad and let it meet the line mnp in q; then $abcd$

is a quadrilateral figure, three of whose sides ab, bc, cd pass through three fixed points in the same straight line; hence the fourth side ad will pass through a fixed point in the same right line.

Now when mab changes its position until the points a, d become coincident, the quadrilateral figure $abcd$ becomes a triangle, and qda becomes a tangent from the point q. Hence from the point q draw the tangent qB (Appendix II. Art. 66); join mBA, ACn, BC; and ABC will be the triangle required.

Similarly, if qB' be the second tangent drawn from q, a second triangle $A'B'C'$ will be determined whose three sides pass through the three points m, n, p.

If the point q falls within the conic section the problem is impossible.

9. When two sides of a triangle inscribed in a conic section pass through two fixed points, the third side will always touch a curve of the second order.

Let the two sides AB, AC pass through the points a_3, b_3; a_2, b_2; then from equations (1), (2) Art. 1, we have

$$m_3 t_1 t_2 - (t_1 + t_2) + n_3 = 0;$$

$$m_2 t_1 t_3 - (t_1 + t_3) + n_2 = 0;$$

and the equation to BC is

$$(a + x)\, t_2 t_3 - \frac{ay}{b}(t_2 + t_3) + (a - x) = 0;$$

$$\text{hence } t_2 = \frac{t_1 - n_3}{m_3 t_1 - 1}, \quad t_3 = \frac{t_1 - n_2}{m_2 t_1 - 1};$$

$$\text{or } t_2 + t_3 = \frac{(m_2 + m_3)\, t_1^2 - (2 + m_2 n_3 + n_2 m_3)\, t_1 + n_2 + n_3}{(m_2 t_1 - 1)(m_3 t_1 - 1)};$$

$$t_2 t_3 = \frac{t_1^2 - (n_2 + n_3)\, t_1 + n_2 n_3}{(m_2 t_1 - 1)(m_3 t_1 - 1)};$$

and substituting these values in the equation to BC,

$$(a + x)\left\{t_1^2 - (n_2 + n_3)\, t_1 + n_2 n_3\right\}$$
$$- \frac{a}{b} y \left\{(m_2 + m_3)\, t_1^2 - (2 + m_2 n_3 + n_2 m_3)\, t_1 + n_2 + n_3\right\}$$
$$+ (a - x)\left\{m_2 m_3 t_1^2 - (m_2 + m_3)\, t_1 + 1\right\} = 0;$$

$$\text{or} \quad \left\{(1 + m_2 m_3)\,a + (1 - m_2 m_3)\,x - \frac{a}{b}(m_2 + m_3)\,y\right\} t_1^2$$

$$+ \left[\left\{m_2 + m_3 - (n_2 + n_3)\right\} x + \frac{a}{b}(2 + m_2 n_3 + n_2 m_3)\,y - (m_2 + m_3 + n_2 + n_3)\,a\right] t_1$$

$$+ (1 + n_2 n_3)\,a - (1 - n_2 n_3)\,x - \frac{a}{b}(n_2 + n_3)\,y = 0;$$

$$\text{or} \quad u t_1^2 + v t_1 + w = 0;$$

where u, v, w are known linear functions of x and y; and to find the curve to which this line is always a tangent, we must make $2ut_1 + v = 0$; $\therefore\ v^2 = 4uw$; which is the equation to a conic section having two tangents $u = 0$, $w = 0$; and $v = 0$ for the equation to the straight line joining their points of contact.

The same proof will equally apply to the hyperbola and parabola.

10. If a polygon of n sides be inscribed in a conic section, and $(n-1)$ sides taken in order pass through $n-1$ fixed points, the remaining side will always touch a curve of the second order.

Let AB, BC, CD, &c. pass through the points a_1, b_1; a_2, b_2; ... a_{n-1}, b_{n-1}; then we have

$$m_1 t_1 t_2 - (t_1 + t_2) + n_1 = 0,$$

$$m_2 t_2 t_3 - (t_2 + t_3) + n_2 = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$m_{n-1} t_{n-1} t_n - (t_{n-1} + t_n) + n_{n-1} = 0;$$

and the equation to the last side is

$$(a + x)\,t_1 t_n - \frac{a}{b}\,y\,(t_1 + t_n) + (a - x) = 0; \qquad (1)$$

$$\therefore\ t_2 = \frac{t_1 - n_1}{m_1 t_1 - 1}, \quad t_3 = \frac{t_2 - n_2}{m_2 t_2 - 1};$$

$$\text{or} \quad t_3 = \frac{(1 - m_1 n_2)\,t_1 + (n_2 - n_1)}{(m_2 - m_1)\,t_1 + 1 - m_2 n_1} = \frac{\alpha_3 t_1 + \beta_3}{\gamma_3 t_1 + \delta_3};$$

where α_3, β_3, γ_3, δ_3 are known quantities.

Similarly, $$t_4 = \frac{t_3 - n_3}{m_3 t_3 - 1} = \frac{\alpha_4 t_1 + \beta_4}{\gamma_4 t_1 + \delta_4};$$

where α_4, β_4, γ_4, δ_4 are known; and

$$t_n = \frac{\alpha_n t_1 + \beta_n}{\gamma_n t_1 + \delta_n};$$

where α_n, β_n, γ_n, δ_n depend only upon a_1, b_1; a_2, b_2; $\ldots a_{n-1}$, b_{n-1}; and are therefore known; but from equation (1)

$$\left\{\left(1 + \frac{x}{a}\right) t_1 - \frac{y}{b}\right\} t_n + 1 - \frac{x}{a} - \frac{y}{b} t_1 = 0;$$

$$\therefore \left\{\left(1 + \frac{x}{a}\right) t_1 - \frac{y}{b}\right\} (\alpha_n t_1 + \beta_n) + \left(1 - \frac{x}{a} - \frac{y}{b} t_1\right) (\gamma_n t_1 + \delta_n) = 0;$$

$$\text{or } u t_1^2 + v t_1 + w = 0;$$

where u, v, w are known linear functions of x and y; hence as before $v^2 = 4uw$ is the equation to the curve which is always touched by the last side of the polygon, and is a curve of the second order.

11. If two sides of a triangle inscribed in a conic section pass through two fixed points, find the condition that the third side may pass through a fixed point.

$$m_3 t_1 t_2 - (t_1 + t_2) + n_3 = 0,$$

$$m_1 t_2 t_3 - (t_2 + t_3) + n_1 = 0;$$

$$\therefore t_2 = \frac{t_1 - n_3}{m_3 t_1 - 1}, \text{ and } (m_1 t_3 - 1) t_2 - (t_3 - n_1) = 0;$$

$$\text{or } (m_1 t_3 - 1)(t_1 - n_3) - (m_3 t_1 - 1)(t_3 - n_1) = 0;$$

$$\therefore (m_1 - m_3) t_1 t_3 + (n_1 m_3 - 1) t_1 + (1 - n_3 m_1) t_3 + n_3 - n_1 = 0;$$

but in order that this may pass through a fixed point a_2, b_2, we have

$$(a + a_2) t_1 t_3 - \frac{a}{b} b_2 (t_1 + t_3) + a - a_2 = 0;$$

or the coefficients of t_1 and t_3 must be equal;

$$\therefore n_1 m_3 - 1 = 1 - n_3 m_1, \text{ or } n_1 m_3 + n_3 m_1 = 2.$$

$$\text{Now}\quad m_1 = \frac{b}{a} \cdot \frac{a + a_1}{b_1}, \quad n_1 = \frac{b}{a} \cdot \frac{a - a_1}{b_1},$$

$$m_3 = \frac{b}{a} \cdot \frac{a + a_3}{b_3}, \quad n_3 = \frac{b}{a} \cdot \frac{a - a_3}{b_3};$$

$$\text{hence}\quad \left(\frac{b}{a}\right)^2 \left(\frac{a + a_1}{b_1} \cdot \frac{a - a_3}{b_3} + \frac{a + a_3}{b_3} \cdot \frac{a - a_1}{b_1}\right) = 2;$$

$$\therefore\ 2\left(\frac{b}{a}\right)^2 \left(\frac{a^2 - a_1 a_3}{b_1 b_3}\right) = 2;$$

$$\therefore\ \frac{a_1 a_3}{a^2} + \frac{b_1 b_3}{b^2} = 1. \quad (1)$$

If two tangents be drawn from a point whose co-ordinates are a_1, b_1; a_3, b_3 will be co-ordinates of any point in the chord of contact; and vice versâ.

12. To find the position of the point through which the third side passes.

Let ab (fig. 161) be a given straight line; and when pairs of tangents are drawn from any point in this line, let the chords of contact meet in the point c; then if b be any given point in ab, and AB, AC be drawn through c, b respectively; the remaining side BC will pass through a fixed point. To find the position of this fixed point, join bc; then if the chord bAC be made to coincide with $bA'B'$, the points BB' will coincide and CB becomes a tangent at B'. Similarly, if A be made to coincide with B', C will coincide with A', and BC becomes the tangent at A'; hence the point a is the point of intersection of the tangents at A', and B'; and lies in the straight line ab.

13. If three sides of a quadrilateral figure inscribed in a curve of the second order pass through three fixed points, find the condition that the fourth side may pass through a fixed point.

Using the notation of Art. 10,

$$t_2 = \frac{t_1 - n_1}{m_1 t_1 - 1}; \quad (m_2 t_3 - 1)\, t_2 - (t_3 - n_2) = 0;$$

$$(m_3 t_4 - 1)\, t_3 - (t_4 - n_3) = 0,$$

$$(m_2 - m_1)\, t_1 t_3 + (n_2 m_1 - 1)\, t_1 + (1 - n_1 m_2)\, t_3 + n_1 - n_2 = 0,$$

$$\text{or } t_3 \left\{(m_2 - m_1)\, t_1 + (1 - n_1 m_2)\right\} + (n_2 m_1 - 1)\, t_1 + (n_1 - n_2) = 0;$$

$$\therefore\ (m_3 t_4 - 1)\left\{(1 - m_1 n_2)\, t_1 + n_2 - n_1\right\}$$
$$- (t_4 - n_3)\left\{(m_2 - m_1)\, t_1 + (1 - n_1 m_2)\right\} = 0,$$

and when the fourth side passes through a fixed point the coefficients of t_1 and t_4 are equal,

$$\text{hence } m_3 (n_2 - n_1) - (1 - n_1 m_2) = n_3 (m_2 - m_1) - (1 - m_1 n_2),$$

$$\text{or } m_1 (n_2 - n_3) + m_2 (n_3 - n_1) + m_3 (n_1 - n_2) = 0;$$

$$\text{Now} \quad n_1 m_2 - m_1 n_2 = \frac{2b^2}{a}\left(\frac{a_2 - a_1}{b_1 b_2}\right);$$

$$\therefore\ \frac{2b^2}{a}\left(\frac{a_2 - a_1}{b_1 b_2} + \frac{a_3 - a_2}{b_3 b_2} + \frac{a_1 - a_3}{b_1 b_3}\right) = 0;$$

$$\text{hence } (a_2 - a_1)\, b_3 + (a_3 - a_2)\, b_1 + (a_1 - a_3)\, b_2 = 0;$$

$$\therefore\ (a_3 - a_2)\, b_1 + (a_1 - a_3)\, b_2 - \left\{(a_3 - a_2) + (a_1 - a_3)\right\} b_3 = 0;$$

$$\therefore\ (a_3 - a_2)(b_1 - b_3) = (a_1 - a_3)(b_3 - b_2);$$

$$\text{or } \frac{b_3 - b_2}{b_3 - b_1} = \frac{a_3 - a_2}{a_3 - a_1}:$$

which shews that the three points must be in the same right line.

14. If $n - 2$ sides taken in order of a polygon of n sides inscribed in a conic section pass through $n - 2$ fixed points; find the position of the point through which the $n - 1^{\text{th}}$ side must pass, in order that the remaining side may pass through a fixed point.

Using the notation of Art. 10;

$$t_{n-1} = \frac{\alpha_{n-1} t_1 + \beta_{n-1}}{\gamma_{n-1} t_1 + \delta_{n-1}}, \qquad (1)$$

where α_{n-1}, β_{n-1}, γ_{n-1}, δ_{n-1} are constant, since they only

depend upon the co-ordinates of the points through which the first $n-2$ sides pass; and

$$(m_{n-1}t_n - 1)\, t_{n-1} = t_n - n_{n-1}; \quad (2)$$

for convenience we will represent equations (1), (2), by

$$t_{n-1} = \frac{At_1 + B}{Ct_1 + D}; \qquad \text{and} \quad (\mu t_n - 1)\, t_{n-1} = (t_n - \nu);$$

$$\therefore\; (\mu t_n - 1)(At_1 + B) = (t_n - \nu)(Ct_1 + D);$$

$$\text{or}\quad (A\mu - C)t_1t_n - (D - B\mu)t_n - (A - C\nu)t_1 + (D\nu - B) = 0;$$

and in order that the n^{th} side may pass through a fixed point the coefficients of t_1 and t_n must be equal;

$$\therefore\; D - B\mu = A - C\nu;$$

and if a_{n-1}, b_{n-1} be the co-ordinates of the point through which the $n-1^{\text{th}}$ side passes,

$$\mu = \frac{b}{a}\left(\frac{a + a_{n-1}}{b_{n-1}}\right); \quad \nu = \frac{b}{a}\left(\frac{a - a_{n-1}}{b_{n-1}}\right),$$

$$\text{or}\quad \frac{a}{b}(D - A)\, b_{n-1} = B(a + a_{n-1}) - C(a - a_{n-1}),$$

$$\text{or}\quad (B + C)\, a_{n-1} = \frac{a}{b}(D - A)\, b_{n-1} - (B - C)\, a;$$

hence a_{n-1}, b_{n-1} lies in a straight line whose equation is

$$(B + C)\, x - \frac{a}{b}(D - A)\, y + (B - C)\, a = 0; \quad (3)$$

and if the $(n-1)^{\text{th}}$ side pass through any point in this line, the n^{th} side will pass through a corresponding fixed point.

15. To find the position of the fixed point through which the n^{th} side passes.

Writing for convenience μ', ν' instead of m_n, n_n, we have

$$\mu' t_1 t_n - (t_1 + t_n) + \nu' = 0;$$

and making this equation coincide with equation (2) Art. 14;

$$\frac{A\mu - C}{D - B\mu} = \mu'; \quad \frac{D\nu - B}{D - B\mu} = \nu'; \quad \text{and } D - B\mu = A - C\nu;$$

$$\therefore\ B\mu' - C\nu' = \frac{AB\mu - DC\nu}{D - B\mu}$$

$$= \frac{AB\mu - D(A - D + B\mu)}{D - B\mu} = D - A;$$

or $D - B\mu' = A - C\nu'$; and as before,

$$(B + C)\,a_n = \frac{a}{b}(D - A)\,b_n - (B - C)\,a;$$

hence a_n, b_n is a point in the straight line in which a_{n-1}, b_{n-1} lies.

Upon the whole we conclude that if a rectilineal figure of n sides be inscribed in a curve of the second order, and $n - 2$ sides taken in order pass through $n - 2$ fixed points, there is a certain straight line, through any *fixed* point of which if the $(n - 1)^{\text{th}}$ side be made to pass, the remaining side will also pass through a fixed point which will be in the same straight line.

In Arts. 11 and 13 the particular cases have been proved when the inscribed figures contain three and four sides respectively.

If $(n - 2)$ sides taken in order pass through $n - 2$ fixed points, and the $n - 1^{\text{th}}$ side passes through a fixed point which does not lie in the straight line determined in equation (3) Art. 14, the n^{th} side will always touch a curve of the second order.

16. If a hexagon be inscribed in a curve of the second order, and five sides taken in order pass through five fixed points in a straight line, the sixth side will pass through a fixed point in the same straight line.

For the hexagon $ABCDEF$ may be divided into two quadrilateral figures; and since AB, BC, CD pass through three fixed points in a straight line, DA will pass through a fixed point in the same straight line. Also since AD, DE, EF pass through three fixed points in a straight line, the remaining side FA will pass through a fixed point in the same straight line.

17. Hence generally, if $2n-1$ sides taken in order of a polygon of $2n$ sides inscribed in a conic section pass through $2n-1$ fixed points in a straight line, the remaining side will pass through a fixed point in the same straight line.

18. If $2n-1$ sides of a polygon $A_1A_2 \ldots A_{2n+1}$ of $2n+1$ sides inscribed in a curve of the second order pass through $2n-1$ fixed points in a given straight line AB, and the $2n^{\text{th}}$ side also pass through O the point of intersection of the chords of contact when pairs of tangents are drawn from any point in AB, the $2n+1^{\text{th}}$ side will pass through a fixed point in the straight line AB.

The chord A_1A_{2n} will pass through some fixed point A in the given straight line AB (Art. 17); and if $A_{2n}A_{2n+1}$ pass through any point in the chord of contact of pairs of tangents drawn from A, the side A_1A_{2n+1} will pass through a fixed point in the same chord of contact (Art. 11); and if it passes through O, the straight line A_1A_{2n+1} will pass through a fixed point in AB (Art. 12), which will be the point of intersection of the chord of contact of a pair of tangents drawn from A with the line AB.

19. By means of Art. 12, a pair of tangents may be drawn at the extremities of a given chord of a conic section.

Produce BA (fig. 162) to any point c; and let DbE be the chord of contact of a pair of tangents drawn from c (Appendix II. Art. 66) intersecting AB in b; through c draw any line $cA'B'$; join $A'bC'$; draw $B'C'E$ meeting DbE in E; then AE, BE are tangents at the points A and B.

If $B'bC_1$ meet the conic section in C_1; $A'C_1$ will meet Db in the same point E.

APPENDIX II.

ON THE GENERAL EQUATION OF THE SECOND DEGREE.

1. To transform the origin to the focus of the curve.

First let the co-ordinates be rectangular, and let

$$ay^2 + bxy + cx^2 + dy + ex + f = \phi(x, y) = 0$$

be the given equation; transform the origin to the point α, β by making

$$x = x' + \alpha, \quad y = y' + \beta;$$

$$\therefore \; a(y' + \beta)^2 + b(x' + \alpha)(y' + \beta) + c(x' + \alpha)^2 + d(y' + \beta) + e(x' + \alpha) + f = 0,$$

$$\text{or} \quad ay'^2 + bx'y' + cx'^2 + d'y' + e'x' + \phi(\alpha, \beta) = 0,$$

$$\text{where } d' = 2a\beta + b\alpha + d; \quad e' = b\beta + 2c\alpha + e.$$

Next, transform the equation to polar co-ordinates by putting

$$x' = \rho\cos\theta, \quad y' = \rho\sin\theta;$$

$$\therefore (a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta)\rho^2 + (d'\sin\theta + e'\cos\theta)\rho + \phi(\alpha, \beta) = 0,$$

$$\text{or} \quad P\rho^2 + Q\rho + \Phi = 0; \quad \therefore \; \frac{1}{\rho} = -\frac{Q \pm \sqrt{Q^2 - 4P\Phi}}{2\Phi};$$

but $Q^2 - 4P\Phi$

$$= (d'^2 - 4a\Phi)\sin^2\theta + (2d'e' - 4b\Phi)\sin\theta\cos\theta + (e'^2 - 4c\Phi)\cos^2\theta;$$

and if α, β be assumed so as to satisfy the conditions

$$d'^2 - 4a\Phi = e'^2 - 4c\Phi; \quad 2d'e' - 4b\Phi = 0;$$

$$\text{then} \quad Q^2 - 4P\Phi = e'^2 - 4c\Phi,$$

$$\text{and} \quad \frac{1}{\rho} = \pm\frac{\sqrt{e'^2 - 4c\Phi}}{2\Phi}\left\{1 \mp \frac{1}{\sqrt{e'^2 - 4c\Phi}}(e'\cos\theta + d'\sin\theta)\right\}$$

$$= \pm\frac{\sqrt{e'^2 - 4c\Phi}}{2\Phi}\left\{1 \mp \frac{e'\sec\delta}{\sqrt{e'^2 - 4c\Phi}}\cos(\theta - \delta)\right\} \text{ if } \frac{d'}{e'} = \tan\delta = \mu \ldots (1)$$

which is the equation to a conic section from the focus, having its axis-major inclined at an angle δ to the axis of x,

$$\text{its eccentricity } E = \frac{e' \sec \delta}{\sqrt{e'^2 - 4c\Phi}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{and semi-latus-rectum } L = \frac{2\Phi}{\sqrt{e'^2 - 4c\Phi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

the two signs corresponding to the equations from the two foci respectively.

Now $d'^2 - e'^2 = 4(a-c)\Phi$, and $d'e' = 2b\Phi$;

$$\therefore \frac{d'}{e'} - \frac{e'}{d'} = \frac{2(a-c)}{b} \text{ or } \mu - \frac{1}{\mu} = -2\cot 2\delta = \frac{2(a-c)}{b};$$

$$\therefore \cot 2\delta = \frac{c-a}{b}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\mu = \frac{a - c \pm \sqrt{(a-c)^2 + b^2}}{b}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\mu + \frac{1}{\mu} = \pm \frac{2\sqrt{(a-c)^2 + b^2}}{b}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$E = \frac{e' \sec \delta}{\sqrt{e'^2 - 4c\Phi}} = \sqrt{\frac{\mu^2 + 1}{1 - \frac{2c}{b}\mu}}; \quad \ldots\ldots\ldots (7)$$

$$\text{and } L = \frac{2\Phi}{\sqrt{e'^2 - 4c\Phi}} = \frac{d'e'}{b\sqrt{e'^2 - 4c\Phi}} = \frac{\mu e'}{b\sqrt{1 - \frac{2c}{b}\mu}}. \quad (8)$$

2. To find α, β and L.

$$\begin{aligned} d'e' - 2b\Phi &= (2a\beta + b\alpha + d)(b\beta + 2c\alpha + e) \\ &\qquad - 2b(a\beta^2 + b\alpha\beta + c\alpha^2 + d\beta + e\alpha + f) \\ &= 2ab\beta^2 + (b^2 + 4ac)\alpha\beta + 2bc\alpha^2 + (2cd + be)\alpha \\ &\quad + (2ae + bd)\beta + de - 2b(a\beta^2 + b\alpha\beta + c\alpha^2 + e\alpha + d\beta + f) \\ &= -(b^2 - 4ac)\alpha\beta + (2cd - be)\alpha + (2ae - bd)\beta - (2bf - de) = 0; \end{aligned}$$

$$\text{or } a\beta - ka - h\beta + g = 0 \;\ldots\ldots\ldots\ldots\ldots \quad (9)$$

where $h = \dfrac{2ae - bd}{b^2 - 4ac}$; $k = \dfrac{2cd - be}{b^2 - 4ac}$ and $g = \dfrac{2bf - de}{b^2 - 4ac}$.

Also $d'^2 - 4a\Phi$

$$= (2a\beta + ba + d)^2 - 4a\,(a\beta^2 + ba\beta + ca^2 + d\beta + ea + f),$$

$$\text{or } d'^2 - 4a\Phi = (b^2 - 4ac)\,a^2 - (4ae - 2bd)\,a + d^2 - 4af.$$

Similarly, $e'^2 - 4c\Phi = (b^2 - 4ac)\beta^2 - (4cd - 2be)\beta + e^2 - 4cf$;

and putting $b^2 - 4ac = m$,

$$d'^2 - 4a\Phi = m\left\{(a - h)^2 + \frac{2a}{b}(hk - g)\right\};$$

$$e'^2 - 4c\Phi = m\left\{(\beta - k)^2 + \frac{2c}{b}(hk - g)\right\};$$

$$\therefore\; (a - h)^2 + \frac{2a}{b}(hk - g) = (\beta - k)^2 + \frac{2c}{b}(hk - g),$$

$$\text{or } (a - h)^2 - (\beta - k)^2 = -\frac{2(a - c)}{b}(hk - g); \quad (10)$$

$$\text{and } (a - h)(\beta - k) = hk - g; \text{ from } (9)$$

$$\text{hence } \frac{a - h}{\beta - k} - \frac{\beta - k}{a - h} = -\frac{2(a - c)}{b} = \frac{1}{\mu} - \mu;$$

$$\therefore\; \frac{\beta - k}{a - h} = \mu, \text{ and } \mu\,(a - h)^2 = hk - g;$$

$$\left.\begin{aligned} \therefore\; a &= h \pm \sqrt{\frac{1}{\mu}(hk - g)} \\ \beta &= k \pm \sqrt{\mu\,(hk - g)} \end{aligned}\right\}. \quad (11)$$

Hence $hk - g$ and μ must have the same sign, and since

$$\mu^2 - \frac{2(a - c)}{b}\mu - 1 = 0,$$

the two values of μ have different signs; and only one of them will make α and β possible.

It will afterwards be proved that in the ellipse where m is negative, $b\left(\mu+\frac{1}{\mu}\right)$ is negative; and the negative sign alone of equation (6) can be used.

Hence in the ellipse

$$\mu+\frac{1}{\mu}=-\frac{2\sqrt{(a-c)^2+b^2}}{b}=-\frac{2\sqrt{(a+c)^2+m}}{b},$$

$$\mu-\frac{1}{\mu}=\frac{2(a-c)}{b};$$

$$\therefore\ \frac{1}{\mu}=-\left\{\frac{a-c+\sqrt{(a+c)^2+m}}{b}\right\},$$

$$\text{and } \frac{1}{\mu}-\frac{2c}{b}=-\left\{\frac{a+c+\sqrt{(a+c)^2+m}}{b}\right\};$$

$$\therefore\ E^2=\frac{\mu+\frac{1}{\mu}}{\frac{1}{\mu}-\frac{2c}{b}}=\frac{2\sqrt{(a+c)^2+m}}{a+c+\sqrt{(a+c)^2+m}}=\frac{2\sqrt{a'^2+m}}{a'+\sqrt{a'^2+m}},$$

where $a+c=a'$, or

$$E^2=\frac{2(a'^2+m)-2a'\sqrt{a'^2+m}}{m}=1+\frac{(a'-\sqrt{a'^2+m})^2}{m}. \quad (12)$$

$$A^2E^2=(\alpha-h)^2+(\beta-k)^2=\left(\mu+\frac{1}{\mu}\right)(hk-g)=-2\sqrt{a'^2+m}\left(\frac{hk-g}{b}\right). (13)$$

$$A^2=-2\sqrt{a'^2+m}\left(\frac{hk-g}{b}\right)\times\frac{1}{E^2}=-(a'+\sqrt{a'^2+m})\left(\frac{hk-g}{b}\right). (14)$$

$$B^2 = A^2(1 - E^2) = -(a' - \sqrt{a'^2 + m})\left(\frac{hk - g}{b}\right). \quad (15)$$

$$L^2 = A^2(1 - E^2)^2 = \frac{(a' - \sqrt{a'^2 + m})^3}{m}\left(\frac{hk - g}{b}\right). \quad (16)$$

And the co-ordinates α_1, β_1 of the extremities of the axis-major are found from the equations

$$(\alpha_1 - h)^2 = \frac{1}{E^2}(\alpha - h)^2 = \frac{a' + \sqrt{a'^2 + m}}{2\sqrt{a'^2 + m}}\left(\frac{hk - g}{\mu}\right). \quad (17)$$

$$(\beta_1 - k)^2 = \frac{1}{E^2}(\beta - k)^2 = \frac{a' + \sqrt{a'^2 + m}}{2\sqrt{a'^2 + m}}\{\mu(hk - g)\}. \quad (18)$$

If α', α''; β', β'' be the two values of α and β respectively determined from equations (11), $\alpha' + \alpha'' = 2h$; $\beta' + \beta'' = 2k$; but α', β'; α'', β'' are the co-ordinates of the two foci; therefore h, k are the co-ordinates of the centre.

3. To find the values of d', e', and $\phi(\alpha, \beta)$.

$$e' = (b\beta + 2c\alpha + e)$$

$$= (bk + 2ch + e) + \left(b\sqrt{\mu} + \frac{2c}{\sqrt{\mu}}\right)\sqrt{hk - g};$$

$$\text{or } e' = (b\mu + 2c)\sqrt{\frac{hk - g}{\mu}}.$$

$$d' = (2a\mu + b)\sqrt{\frac{hk - g}{\mu}}.$$

$$\text{And } 2b\phi(\alpha, \beta) = d'e' = \mu e'^2 = (b\mu + 2c)^2(hk - g).$$

4. To deduce from the above expressions the co-ordinates of the vertex, and latus rectum when $b^2 - 4ac = 0$.

In this case the values of h and k become infinite;

$$\text{let } 2cd - be = K,\ 2ae - bd = H,\ 2bf - de = G;$$

$$\text{then } \mu = \frac{(a - c) - \sqrt{(a + c)^2 + m}}{b},$$

$$\frac{1}{\mu} = -\left\{\frac{(a-c)+\sqrt{(a+c)^2+m}}{b}\right\};$$

and expanding to the first power of m,

$$\mu = -\frac{2c}{b} - \frac{m}{2a'b} = -\frac{2c}{b}\left(1+\frac{m}{4a'c}\right),$$

$$\frac{1}{\mu} = -\frac{2a}{b} - \frac{m}{2a'b} = -\frac{2a}{b}\left(1+\frac{m}{4a'a}\right);$$

$$\text{now } hk - g = -2b\,\frac{(ae^2 - bde + cd^2 + mf)}{m^2};$$

and if $M = d^2 - 4af$, $N = e^2 - 4cf$;

$$\frac{mh^2 - M}{m} = 4a\,\frac{(ae^2 - bde + cd^2 + mf)}{m^2};$$

$$\frac{mk^2 - N}{m} = 4c\,\frac{(ae^2 - bde + cd^2 + mf)}{m^2};$$

$$\therefore\ -\frac{hk-g}{2b} = \frac{mh^2 - M}{4am} = \frac{mk^2 - N}{4cm};$$

$$\text{hence } \alpha = \frac{H}{m} - \frac{1}{m}\sqrt{(H^2 - mM)\left(1+\frac{m}{4aa'}\right)}$$

$$\text{or } \alpha = \frac{H}{m} - \frac{H}{m}\left(1+\frac{m}{8aa'} - \frac{mM}{2H^2}\right) = \frac{M}{2H} - \frac{H}{8aa'}; \qquad (19)$$

$$\beta = \frac{K}{m} - \frac{1}{m}\sqrt{(K^2 - mN)\left(1+\frac{m}{4ca'}\right)} = \frac{N}{2K} - \frac{K}{8ca'} \ldots\ (20)$$

$$L^2 = \frac{m^2}{8a'^3} \times \left(\frac{K^2 - mN}{2cm^2}\right) = \frac{K^2}{16ca'^3}\left(1 - \frac{mN}{K^2}\right),$$

$$\text{or } L = \pm\frac{K}{4\sqrt{c}\,a'^{\frac{3}{2}}}\left(1 - \frac{mN}{2K^2}\right); \ \ldots\ldots\ldots\ldots\ (a)$$

$$\text{and when } m \text{ vanishes } L = \pm\frac{K}{4\sqrt{ca'^3}}. \ \ldots\ldots\ldots\ldots\ (21)$$

$$E^2 = 1 + \frac{m}{4a'^2}; \quad \therefore \; E = 1 + \frac{m}{8a'^2} \; \ldots\ldots\ldots\ldots\ldots (\beta)$$

which gives the eccentricity of the ellipse when m becomes very small.

Also $\alpha_1 = h + \frac{1}{E}(\alpha - h) = \frac{H}{m} - \frac{H}{m}\left(1 + \frac{m}{8aa'} - \frac{mM}{2H^2}\right)\left(1 - \frac{m}{8a'^2}\right)$,

$$\text{or } \alpha_1 = \frac{M}{2H} - \frac{H(a' - a)}{8aa'^2} = \frac{M}{2H} - \frac{cH}{8aa'^2} \; \ldots\ldots\ldots \; (22)$$

Similarly, $\beta_1 = k + \frac{1}{E}(\beta - k)$;

$$\text{or } \beta_1 = \frac{K}{m} - \frac{K}{m}\left(1 + \frac{m}{8ca'} - \frac{mN}{2K^2}\right)\left(1 - \frac{m}{8a'^2}\right) = \frac{N}{2K} - \frac{aK}{8ca'^2} \; (23).$$

5. When $b^2 - 4ac = 0$, to determine independently the latus rectum, the position of the axis, and the co-ordinates of the vertex of the parabola.

From equation (9)

$$(2cd - be)\,\alpha + (2ae - bd)\,\beta - (2bf - de) = 0;$$

or $K\alpha + H\beta = G$; $\therefore \; \beta + \frac{K}{H}\alpha = \frac{G}{H}$, or $\beta - \frac{b}{2a}\alpha = \frac{G}{H}$; (A)

also the two values of μ, derived from the equation

$$\mu - \frac{1}{\mu} = \frac{2(a - c)}{b}, \text{ are } \frac{2a}{b} \text{ and } -\frac{2c}{b};$$

$$\text{but } \mu = \frac{d'}{e'} = \frac{2a\beta + b\alpha + d}{b\beta + 2c\alpha + e} = \frac{2a}{b} - \frac{(2ae - bd)}{b(b\beta + 2c\alpha + e)}$$

and cannot therefore $= \frac{2a}{b}$;

hence the only admissible value of μ is $-\frac{2c}{b} = -\frac{b}{2a}$.

Again, $d'^2 - 4a\Phi = -(2H\alpha - M)$; and $e'^2 - 4c\Phi = -(2K\beta - N)$;

$$\therefore \; 2H\alpha - M = 2K\beta - N; \; \ldots\ldots\ldots\ldots\ldots\ldots \; (B)$$

and the intersection of the two straight lines represented by equations (A) and (B) will be in the focus of the parabola.

$$\text{Also } \frac{K}{2H}(2H\alpha) + \frac{H}{2K}(2K\beta) = G;$$

$$\therefore \frac{K}{2H}(2H\alpha - M) + \frac{H}{2K}(2K\beta - N) = G - \frac{K}{2H}M - \frac{H}{2K}N;$$

$$\text{hence } \left(\frac{K}{2H} + \frac{H}{2K}\right)(2K\beta - N) = G - \frac{K}{2H}.M - \frac{H}{2K}.N;$$

$$\text{or } -\left(\frac{a+c}{b}\right)(2K\beta - N) = 2bf - de + \frac{c}{b}(d^2 - 4af) + \frac{a}{b}(e^2 - 4cf)$$

$$= \frac{cd^2 + ae^2 - bde}{b} = \frac{K^2}{4bc} = \frac{H^2}{4ab};$$

$$\therefore 2K\beta - N = 2H\alpha - M = -\frac{K^2}{4c(a+c)} = -\frac{H^2}{4a(a+c)};$$

$$\therefore \beta = \frac{N}{2K} - \frac{K}{8c(a+c)} \ldots\ldots\ldots\ldots (C)$$

$$\alpha = \frac{M}{2H} - \frac{H}{8a(a+c)} \ldots\ldots\ldots (D)$$

$$\text{Again, } \frac{d'}{e'} = \frac{2a\beta + b\alpha + d}{b\beta + 2c\alpha + e} = \mu = -\frac{b}{2a};$$

$$\therefore (4a^2 + b^2)\beta + 2b(a+c)\alpha + 2ad + be = 0;$$

$$\text{or } \beta + \frac{b}{2a}\alpha + \frac{2ad + be}{4a(a+c)} = 0, \ldots\ldots\ldots\ldots\ldots (E)$$

$$\text{and } e' = b\left(\beta + \frac{b}{2a}\alpha\right) + e = e - \frac{c}{b}\left(\frac{2ad + be}{a+c}\right) = \frac{H}{2(a+c)}; \text{ (F)}$$

$$\text{but } d'e' = 2b\Phi = \mu e'^2 = -\frac{2c}{b}e'^2; \ \therefore e'^2 = -4a\Phi;$$

$$\text{and } \frac{L}{E} = \frac{2\Phi\cos\delta}{e'} = -\frac{e'\cos\delta}{2a} = -\frac{H\cos\delta}{4a(a+c)};$$

$$\therefore\ L = -\frac{H}{4\sqrt{a}\,(a+c)^{\frac{3}{2}}} = \frac{K}{4\sqrt{c}\,(a+c)^{\frac{3}{2}}} \ldots\ldots\ldots \text{(G)}$$

6. To find the equation to the axis of the parabola.

Since $\tan\delta = -\frac{b}{2a}$, equation (E) is the equation to the principal diameter.

7. When the general equation of the second degree is referred to oblique axes, to find the polar equation from the focus.

Let $ay^2 + bxy + cx^2 + dy + ex + f = \phi(x, y) = 0$, be the equation of the second degree referred to two axes inclined to each other at an angle ω; transform the origin to a point α, β by making $x = x' + \alpha$, $y = y' + \beta$;

$$\therefore\ ay'^2 + bx'y' + cx'^2 + d'y' + e'x' + \phi(\alpha, \beta) = 0,$$

$$\text{where } d' = 2a\beta + b\alpha + d;\quad e' = b\beta + 2c\alpha + e.$$

Next let the equation be transformed to polar co-ordinates by making

$$x' = \frac{\sin(\omega - \theta)}{\sin\omega}\rho;\quad y' = \frac{\sin\theta}{\sin\omega}\rho;$$

$$\therefore\ \left\{a\sin^2\theta + b\sin\theta\sin(\omega-\theta) + c\sin^2(\omega-\theta)\right\}\rho^2$$

$$+ \sin\omega\left\{d'\sin\theta + e'\sin(\omega-\theta)\right\}\rho + \sin^2\omega\,\phi(\alpha, \beta) = 0,$$

r $\left\{(a - b\cos\omega + c\cos^2\omega)\sin^2\theta + (b\sin\omega - c\sin 2\omega)\sin\theta\cos\theta + c\sin^2\omega\cos^2\theta\right\}\rho^2$

$$+ \sin\omega\left\{(d' - e'\cos\omega)\sin\theta + e'\sin\omega\cos\theta\right\}\rho + \sin^2\omega\,\phi(\alpha, \beta) = 0;$$

and representing this equation by $P\rho^2 + Q\sin\omega\,\rho + \sin^2\omega\,\Phi = 0$, we have

$$\frac{1}{\rho^2} + \frac{Q}{\sin\omega\,\Phi}\,\frac{1}{\rho} = -\frac{P}{\sin^2\omega\,\Phi},\ \text{or}\ \frac{1}{\rho} = \frac{-Q \pm \sqrt{Q^2 - 4P\Phi}}{2\sin\omega\,\Phi}.$$

$$\text{Now } Q^2 - 4P\Phi = \{(d' - e'\cos\omega)\sin\theta + e'\sin\omega\cos\theta\}^2$$
$$- 4\Phi\{(a - b\cos\omega + c\cos^2\omega)\sin^2\theta$$
$$+ (b\sin\omega - c\sin 2\omega)\sin\theta\cos\theta + c\sin^2\omega\cos^2\theta\};$$

and assuming α, β so that the coefficient of $\sin\theta\cos\theta$ may vanish, and the coefficients of $\sin^2\theta$ and $\cos^2\theta$ be equal in the expression for $Q^2 - 4P\Phi$, we have

$$(d' - e'\cos\omega)^2 - 4\Phi(a - b\cos\omega + c\cos^2\omega)$$
$$= e'^2\sin^2\omega - 4c\Phi\sin^2\omega, \qquad (1)$$

$$\text{and } 2(d' - e'\cos\omega)e'\sin\omega - 4\Phi(b\sin\omega - c\sin 2\omega) = 0, \qquad (2)$$

$$\text{or } (d' - e'\cos\omega)e' = 2\Phi(b - 2c\cos\omega), \qquad (3)$$

$$\text{and } d'^2 - 4a\Phi - 2d'e'\cos\omega + e'^2\cos 2\omega$$
$$= 4\Phi(c\cos 2\omega - b\cos\omega) = -4c\Phi - 2(d' - e'\cos\omega)e'\cos\omega;$$
$$\therefore\ d'^2 - 4a\Phi = e'^2 - 4c\Phi, \qquad (4)$$
$$\text{and } d'e' - 2b\Phi = \cos\omega\,(e'^2 - 4c\Phi).$$

$$\text{Hence, putting } \frac{d' - e'\cos\omega}{e'\sin\omega} = \mu = \tan\delta, \qquad (5)$$

$$\frac{1}{\rho} = \frac{-e'\sin\omega\sec\delta\cos(\theta - \delta) \pm \sin\omega\sqrt{e'^2 - 4c\Phi}}{2\sin\omega\,\Phi}$$

$$\text{or } \frac{1}{\rho} = \pm\frac{\sqrt{e'^2 - 4c\Phi} \mp e'\sec\delta\cos(\theta - \delta)}{2\Phi} \qquad (6)$$

which is the polar equation to a conic section from the focus,

$$\text{whose latus-rectum} = \frac{4\Phi}{\sqrt{e'^2 - 4c\Phi}}; \qquad (7)$$

$$\text{eccentricity } E = \frac{e'\sec\delta}{\sqrt{e'^2 - 4c\Phi}}; \qquad (8)$$

and inclination of the axis-major to the prime radius $= \delta$.

Also from (3) $e'(d' - e'\cos\omega) = 2\Phi(b - 2c\cos\omega)$;

$$\therefore\ e'^2\sin\omega\,\mu = 2\Phi(b - 2c\cos\omega),$$

$$\text{and } E = \sqrt{\frac{\mu^2 + 1}{1 - \dfrac{2c \sin\omega}{b - 2c\cos\omega}\mu}} \;.\;\ldots\ldots\ldots\ldots\ldots \;(9)$$

8. To find μ, we have

$$e'^2 \sin\omega\mu = 2(b - 2c\cos\omega)\,\Phi\,;$$

$$\frac{d'}{e'} = \mu\sin\omega + \cos\omega, \text{ and } d'^2 - e'^2 = 4(a - c)\,\Phi\,;$$

$$\therefore\; e'^2\left\{(\mu^2 - 1)\sin^2\omega + \mu\sin 2\omega\right\} = 4(a - c)\,\Phi\,;$$

$$\text{hence } \frac{(\mu^2 - 1)\sin^2\omega + \mu\sin 2\omega}{\mu\sin\omega} = \frac{2(a - c)}{b - 2c\cos\omega},$$

$$\text{or } \mu - \frac{1}{\mu} = \frac{2(a - b\cos\omega + c\cos 2\omega)}{b\sin\omega - c\sin 2\omega} = -2\cot 2\delta, \;\ldots\; (10)$$

$$\text{and } \mu + \frac{1}{\mu} = \pm\sqrt{4 + \left(\mu - \frac{1}{\mu}\right)^2}$$

$$\text{or } \mu + \frac{1}{\mu} = \pm\frac{2\sqrt{(a - b\cos\omega + c)^2 + (b^2 - 4ac)\sin^2\omega}}{b\sin\omega - c\sin 2\omega}. \;(11)$$

It will afterwards be proved that in the ellipse and parabola the negative sign only can be used in the expression for $\mu + \dfrac{1}{\mu}$.

9. To find α, β, and L.

$$\text{From (Art. 2) } d'^2 - 4a\Phi = m\left\{(\alpha - h)^2 + \frac{2a}{b}(hk - g)\right\},$$

$$e'^2 - 4c\Phi = m\left\{(\beta - k)^2 + \frac{2c}{b}(hk - g)\right\},$$

$$d'e' - 2b\Phi = -m\left\{(\alpha - h)(\beta - k) - (hk - g)\right\};$$

but $d'^2 - 4a\Phi = e'^2 - 4c\Phi$, and $d'e' - 2b\Phi = (e'^2 - 4c\Phi)\cos\omega$

$$\therefore\; (\alpha - h)^2 - (\beta - k)^2 = -\frac{2(a - c)}{b}(hk - g), \;\ldots\ldots (12$$

$$\cos\omega\,(\beta - k)^2 + (\alpha - h)(\beta - k) = \frac{b - 2c\cos\omega}{b}(hk - g). \quad (13)$$

Also $\dfrac{\alpha - h}{\beta - k} = \dfrac{\sin(\omega - \delta)}{\sin\delta}$; for dividing (12) by (13) and putting $\dfrac{\alpha - h}{\beta - k} = \lambda$;

$$\therefore \frac{\lambda^2 - 1}{\cos\omega + \lambda} = -\frac{2(a - c)}{b - 2c\cos\omega} = -\left\{\frac{(\mu^2 - 1)\sin\omega + 2\mu\cos\omega}{\mu}\right\},$$

from which $\lambda = \dfrac{\sin\omega}{\mu} - \cos\omega = \dfrac{\sin(\omega - \delta)}{\sin\delta}$;

$$\therefore (\beta - k)^2\left\{\cos\omega + \frac{\sin(\omega - \delta)}{\sin\delta}\right\} = \frac{b - 2c\cos\omega}{b}(hk - g);$$

$$\text{hence } (\beta - k)^2 = \frac{b - 2c\cos\omega}{b\sin\omega}\tan\delta\,(hk - g); \;\ldots\ldots \;(14)$$

$$\text{similarly, } (\alpha - h)^2 = \frac{b - 2a\cos\omega}{b\sin\omega}\tan(\omega - \delta)(hk - g), \quad (15)$$

$$\text{and } (AE)^2 = (\beta - k)^2\frac{\sin^2\omega}{\sin^2\delta} = \frac{2(b\sin\omega - c\sin 2\omega)}{b\sin 2\delta}(hk - g)$$

$$\text{or } (AE^2) = \frac{(b\sin\omega - c\sin 2\omega)}{b}\left(\mu + \frac{1}{\mu}\right)(hk - g). \;\ldots\ldots\; (16)$$

Let $a - b\cos\omega + c = a'$, $b - 2c\cos\omega = b'$, and

$$(b^2 - 4ac)\sin^2\omega = m';$$

therefore in the ellipse

$$\mu + \frac{1}{\mu} = -\frac{2\sqrt{a'^2 + m'}}{b'\sin\omega}, \quad \mu - \frac{1}{\mu} = \frac{2(a' - 2c\sin^2\omega)}{b'\sin\omega},$$

$$\text{or } \frac{1}{\mu} = -\left(\frac{a' - 2c\sin^2\omega + \sqrt{a'^2 + m'}}{b'\sin\omega}\right),$$

$$\text{and } \frac{1}{\mu} - \frac{2c\sin^2\omega}{b'\sin\omega} = -\left(\frac{a' + \sqrt{a'^2 + m'}}{b'\sin\omega}\right);$$

hence $E^2 = \dfrac{2\sqrt{a'^2 + m'}}{a' + \sqrt{a'^2 + m'}} = \dfrac{2(a'^2 + m') - 2a'\sqrt{a'^2 + m'}}{m'}$

$$\text{or } E^2 = 1 + \frac{(a' - \sqrt{a'^2 + m'})^2}{m'}; \quad \ldots \quad (17)$$

$$A^2 = \sin\omega \,.\, b'\left(\mu + \frac{1}{\mu}\right)\left(\frac{hk - g}{b}\right) \times \frac{1}{E^2};$$

$$\therefore\ A^2 = -\,(a' + \sqrt{a'^2 + m'})\left(\frac{hk - g}{b}\right), \ldots\ldots\ldots\ldots (18)$$

$$B^2 = A^2(1 - E^2) = A^2\frac{(a' - \sqrt{a'^2 + m'})}{a' + \sqrt{a'^2 + m'}};$$

$$\therefore\ B^2 = -\,(a' - \sqrt{a'^2 + m'})\left(\frac{hk - g}{b}\right). \ldots (19)$$

$L^2 = A^2(1 - E^2)^2 = B^2(1 - E^2)$;

$$\therefore\ L^2 = \frac{(a' - \sqrt{a'^2 + m'})^3}{m'}\left(\frac{hk - g}{b}\right), \ldots\ldots (20)$$

$$A^2E^2 = -\,2\sqrt{a'^2 + m'}\left(\frac{hk - g}{b}\right), \ldots\ldots\ldots (21)$$

$$(a - h)^2 = \frac{b - 2a\cos\omega}{\sin\omega}\tan(\omega - \delta)\left(\frac{hk - g}{b}\right), \ldots (22)$$

$$(\beta - k)^2 = \frac{b - 2c\cos\omega}{\sin\omega}\tan\delta\left(\frac{hk - g}{b}\right); \quad \ldots\ldots (23)$$

$$\text{or } (\beta - k)^2 = \frac{b'}{\sin\omega}\left(\frac{a' - 2c\sin^2\omega - \sqrt{a'^2 + m'}}{b'\sin\omega}\right)\left(\frac{hk - g}{b}\right);$$

$$\therefore\ (\beta - k)^2 = (a - b\cos\omega + c\cos 2\omega - \sqrt{a'^2 + m'})\left(\frac{hk - g}{b\sin^2\omega}\right);$$

$$\text{and } (a - h)^2 = (c - b\cos\omega + a\cos 2\omega - \sqrt{a'^2 + m'})\left(\frac{hk - g}{b\sin^2\omega}\right).$$

And if a_1, β_1 be the co-ordinates of the extremities of the axis-major;

$$(\beta_1 - k)^2 = \frac{1}{E^2}(\beta - k)^2 = \left(\frac{a' + \sqrt{a'^2 + m'}}{2\sqrt{a'^2 + m'}}\right)(\beta - k)^2$$

$$\text{or } (\beta_1 - k)^2 = -\left\{\frac{m + 2c(a' + \sqrt{a'^2 + m'})}{2\sqrt{a'^2 + m'}}\right\}\left(\frac{hk - g}{b}\right). \quad (24)$$

Similarly,

$$(a_1 - h)^2 = -\left\{\frac{m + 2a(a' + \sqrt{a'^2 + m'})}{2\sqrt{a'^2 + m'}}\right\}\left(\frac{hk - g}{b}\right). \quad (25)$$

And (area of the ellipse)$^2 = \pi^2 A^2 B^2 = -\pi^2 m' \left(\frac{hk - g}{b}\right)^2$;

therefore area of the ellipse $= -\pi\sqrt{4ac - b^2}.\sin\omega\left(\frac{hk - g}{b}\right)$. (26)

It may be observed that as in the former case

$$\frac{hk - g}{b} = -\left(\frac{mh^2 - M}{2am}\right) = -\left(\frac{mk^2 - N}{2cm}\right).$$

10. To deduce from the above the expressions for the position of the axis, the co-ordinates of the vertex, and the latus rectum, when $b^2 - 4ac$ vanishes.

In this case we have $\mu = \dfrac{a' - 2c\sin^2\omega - \sqrt{a'^2 + m'}}{b'\sin\omega}$

$$= \frac{-2c\sin^2\omega - \dfrac{m'}{2a'}}{b'\sin\omega} = -\frac{2c\sin\omega}{b'}\left(1 + \frac{m'}{4a'c\sin^2\omega}\right)$$

or $\mu = -\dfrac{2c\sin\omega}{b'}\left(1 + \dfrac{m}{4a'c}\right)$. .... (1)

Hence $(\beta - k)^2 = -\dfrac{b'\mu}{\sin\omega}\left(\dfrac{mk^2 - N}{2cm}\right)$,

$$\text{or } \beta = \frac{K}{m} \pm \sqrt{\left(1 + \frac{m}{4a'c}\right)\frac{K^2}{m^2}\left(1 - \frac{mN}{K^2}\right)}$$

$$= \frac{K}{m} \pm \frac{K}{m}\left\{1 + m\left(\frac{1}{8a'c} - \frac{N}{2K^2}\right)\right\};$$

$$\text{or } \beta = \frac{N}{2K} - \frac{K}{8a'c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

$$\text{Similarly, } \alpha = \frac{M}{2H} - \frac{H}{8a'a}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

$$\text{Also } \beta_1 - k = \frac{1}{E}(\beta - k), \text{ and } E^2 = 1 + \frac{m'}{4a'^2};$$

$$\text{or } \frac{1}{E} = 1 - \frac{m'}{8a'^2} = 1 - \frac{m\sin^2\omega}{8a'^2};$$

$$\therefore \beta_1 = \frac{K}{m} - \frac{K}{m}\left\{1 + m\left(\frac{1}{8a'c} - \frac{N}{2K^2}\right)\right\}\left(1 - \frac{m\sin^2\omega}{8a'^2}\right)$$

$$= \frac{N}{2K} - \frac{K}{8a'^2c}(a' - c\sin^2\omega)$$

$$= \frac{N}{2K} - \frac{K}{8a'^2c}(a - b\cos\omega + c\cos^2\omega); \quad \ldots\ldots \text{(4)}$$

$$\text{similarly, } \alpha_1 = \frac{M}{2H} - \frac{H}{8a'^2a}(c - b\cos\omega + a\cos^2\omega). \quad \ldots\ldots \text{(5)}$$

And from equation (1) $\mu = -\dfrac{2c\sin\omega}{b'}$;

$$\therefore \sin^2\delta = \frac{c\sin^2\omega}{a'}, \ \cos^2\delta = \frac{a - b\cos\omega + c\cos^2\omega}{a'};$$

$$\therefore \beta_1 = \frac{N}{2K} - \frac{K\cos^2\delta}{8a'c};$$

$$\alpha_1 = \frac{M}{2H} - \frac{H\cos^2(\omega - \delta)}{8a'a}. \quad \ldots\ldots\ldots\ldots\ldots \text{(6)}$$

$$L^2 = -\frac{(a' - \sqrt{a'^2 + m'})^3}{m'}\left(\frac{mk^2 - N}{2cm}\right)$$

$$= \frac{m'^2}{8a'^3}\left(\frac{K^2 - mN}{2cm^2}\right) = \sin^4\omega\left(\frac{K^2 - mN}{16ca'^3}\right);$$

$$\text{or } L = \frac{\sin^2\omega K}{4\sqrt{c}\,a'^{\frac{3}{2}}}\left(1 - \frac{mN}{2K^2}\right) \text{ when } m \text{ is small;}$$

and when m vanishes, $L = \pm\dfrac{\sin^2\omega K}{4\sqrt{c}\,a'^{\frac{3}{2}}}$. .... (7)

When m is very small,

$$\frac{B^2}{A^2} = 1 - E^2 = -\frac{m'}{4a'^2} = \frac{(4ac - b^2)\sin^2\omega}{4a'^2};$$

or as $b^2 - 4ac$ diminishes, the ratio of the axes of the ellipse approaches to

$$\frac{\sqrt{4ac - b^2}\,.\,\sin\omega}{2(a - b\cos\omega + c)}. \quad \text{..................} \quad (8)$$

11. When $b^2 - 4ac = 0$, to determine independently the position of the diameter, the co-ordinates of the vertex, and the latus rectum of the parabola.

From Art. 5,

$$d'^2 - 4a\Phi = -(2H\alpha - M),\ e'^2 - 4c\Phi = -(2K\beta - N);$$

$$\text{and } d'e' - 2b\Phi = K\alpha + H\beta - G;$$

$$\therefore\ 2H\alpha - M = 2K\beta - N, \text{ and } K\alpha + H\beta - G = -(2K\beta - N)\cos\omega;$$

$$\text{or } \frac{K}{2H}(2H\alpha) + \frac{H}{2K}(2K\beta) + (2K\beta - N)\cos\omega = G;$$

$$\therefore\ \left(\frac{K}{2H} + \frac{H}{2K} + \cos\omega\right)(2K\beta - N) = G - \frac{K}{2H}M - \frac{H}{2K}.N;$$

$$\text{hence } -\left(\frac{a+c-b\cos\omega}{b}\right)(2K\beta - N)$$

$$= 2bf - de + \frac{c}{b}(d^2 - 4af) + \frac{a}{b}(e^2 - 4cf)$$

$$= \frac{cd^2 + ae^2 - bde}{b} = \frac{K^2}{4bc} = \frac{H^2}{4ab};$$

$$\therefore\ \beta = \frac{N}{2K} - \frac{K}{8ca'} = \frac{N}{2K} + \frac{H}{4ba'}; \ \ldots\ldots\ldots\ldots \ (1)$$

$$\text{similarly, } \alpha = \frac{M}{2H} - \frac{H}{8aa'} = \frac{M}{2H} + \frac{K}{4ba'}. \ \ldots\ldots\ldots\ldots \ (2)$$

Also $\dfrac{d'}{e'} = (\cot\omega + \mu)\sin\omega$; and since $b^2 - 4ac = 0$;

equations 10 and 11, Art. 8, give

$$\mu = \frac{b - 2c\cos\omega}{2c\sin\omega}, \text{ or } \frac{-2c\sin\omega}{b - 2c\cos\omega};$$

$$\therefore\ \mu + \cot\omega = \frac{b}{2c\sin\omega} \text{ or } \frac{b\cos\omega - 2c}{\sin\omega\,(b - 2c\cos\omega)};$$

$$\text{and } \frac{d'}{e'} = \frac{b}{2c} \text{ or } \frac{b\cos\omega - 2c}{b - 2c\cos\omega};$$

but $\dfrac{d'}{e'} = \dfrac{2a\beta + b\alpha + d}{b\beta + 2c\alpha + e} = \dfrac{b}{2c} + \dfrac{(2cd - be)}{2c\,(b\beta + 2c\alpha + e)} = \dfrac{b}{2c} + \dfrac{K}{2ce'}$,

which cannot $= \dfrac{b}{2c}$; hence the only admissible value of μ is

$$\frac{-2c\sin\omega}{b - 2c\cos\omega}; \text{ and } \sin\delta = \frac{2c\sin\omega}{\sqrt{(b - 2c\cos\omega)^2 + 4c^2\sin^2\omega}}$$

$$= \frac{\sqrt{c}\,\sin\omega}{\sqrt{a'}};$$

$$\text{also } \frac{d'}{e'} = \frac{b}{2c} + \frac{K}{2ce'} = \frac{b\cos\omega - 2c}{b - 2c\cos\omega};$$

$$\therefore\ \frac{K}{2ce'} = \frac{b\cos\omega - 2c}{b - 2c\cos\omega} - \frac{b}{2c} = \frac{4bc\cos\omega - 4ac - 4c^2}{2c\,(b - 2c\cos\omega)} = -\frac{2a'}{b'};$$

$$\text{hence } e' = -\frac{b'K}{4ca'};$$

$$\text{and } L = \frac{L}{E} = \frac{2\Phi\cos\delta}{e'} = \frac{e'^2\sin\omega\tan\delta.\cos\delta}{e'\,(b - 2c\cos\omega)} = \frac{e'\sin\omega\sin\delta}{b - 2c\cos\omega};$$

$$\text{but } \sin\delta = \frac{\sqrt{c}\sin\omega}{\sqrt{a'}};\ \therefore\ L = -\frac{\sin^2\omega\,K}{4\sqrt{c}\,a'^{\frac{3}{2}}} \quad\ldots\ldots\ldots (3)$$

$$\beta_1 = \beta - \frac{L}{2}\frac{\sin\delta}{\sin\omega} = \frac{N}{2K} - \frac{K}{8ca'} + \frac{K\sin^2\omega}{8a'^2}; \quad\ldots\ldots\ldots (4)$$

$$a_1 = a - \frac{L}{2}\frac{\sin(\omega-\delta)}{\sin\omega} = \frac{M}{2H} - \frac{H}{8aa'} + \frac{H\sin^2\omega}{8a'^2} \quad\ldots. (5)$$

$$\text{Again } e' = b\beta + 2ca + e = -\frac{b'K}{4ca'},$$

$$\text{and } \frac{\sin\delta}{\sin(\omega-\delta)} = -\left(\frac{2c}{b}\right);$$

$$\therefore\ b\beta + 2ca + \frac{b'K}{4ca'} + e = 0; \quad\ldots\ldots\ldots\ldots\ldots (6)$$

is the equation to the principal diameter;

and the intersection of the two straight lines

$$2Ha - M = 2K\beta - N;$$

$$\text{and } Ka + H\beta - G = -\cos\omega\,(2K\beta - N) = -\frac{K^2\cos\omega}{4ca'} \quad\ldots. (7)$$

will be in the focus of the parabola.

Let A be the vertex, S the focus, ASa (fig. 163) the axis of a parabola, and let the equation be transformed so that $\angle PSa = \theta - \delta$; then when the axes are inclined to each other at an angle ω,

$$d'e' - 2b\Phi = (e'^2 - 4c\Phi)\cos\omega;$$

$$\therefore\ (d' - e'\cos\omega)\,e' = 2\,(b - 2c\cos\omega)\,\Phi;$$

$$\text{or } \mu e'^2 \sin\omega = 2b'\Phi; \quad \text{but } \mu = -\frac{2c\sin\omega}{b'};$$

$$\therefore \Phi = -\frac{c\sin^2\omega \,.\, e'^2}{b'^2},$$

and is negative since a and c are positive.

Hence in order that the latus rectum of the parabola may be positive we must take the equation

$$\frac{1}{\rho} = -\frac{\sqrt{e'^2 - 4c\Phi}}{2\Phi}\left\{1 + \frac{e'\sec\delta}{\sqrt{e'^2 - 4c\Phi}}\cos(\theta - \delta)\right\};$$

therefore $e'\sec\delta$ is negative;

$$\text{but } e'\tan\delta = \frac{K\sin\omega}{2a'}; \quad \therefore e'\sec\delta = \frac{K\sin\omega}{2a'\sin\delta}$$

which is negative; hence K and $\sin\delta$ have different signs.

From equation (6) if b be positive, CD (fig. 164) is the direction of the axis, and the parabola will assume the position A_2P_2 or A_4P_4 according as $\sin\delta$ is positive or negative; i. e. according as K is negative or positive.

If b be negative, $C'D'$ is the direction of the axis; and the parabola will assume the position A_1P_1 or A_3P_3 according as $\sin\delta$ is positive or negative; i. e. according as K is negative or positive.

This equally applies to Art. 6, where the axes are rectangular.

12. If any relation among the coefficients of the given equation should make d' and $e' = 0$; since $d'^2 - e'^2 = 4(a - c)\Phi$, we have $\Phi = 0$; and the equation when transformed to polar co-ordinates becomes

$$P\rho^2 = 0, \text{ or } \left\{a\sin^2\theta + b\sin\theta\sin(\omega - \theta) + c\sin^2(\omega - \theta)\right\}\rho^2 = 0,$$

and since ρ is not always $= 0$; this equation determines two values of (θ), or the equation represents two straight lines passing through the point α, β.

13. When d and e are each $= 0$, the origin is the centre;

$$\text{and } d'e' - 2b\Phi = -m\alpha\beta - 2bf,$$

$$d'^2 - 4a\Phi = m\alpha^2 - 4af, \quad e'^2 - 4c\Phi = m\beta^2 - 4cf;$$

$$\therefore\ m(\alpha^2 - \beta^2) = 4(a-c)f, \text{ and } \cos\omega\,(m\beta^2 - 4cf) = -m\alpha\beta - 2bf;$$

$$\therefore\ A^2 = \frac{2f}{m}\left(a' + \sqrt{a'^2 + m'}\right),$$

$$B^2 = \frac{2f}{m}\left(a' - \sqrt{a'^2 + m'}\right),$$

$$\alpha^2 = -\frac{(b - 2a\cos\omega)}{b\sin\omega}\tan(\omega - \delta)\frac{2bf}{m}$$

$$= -\frac{2(b - 2a\cos\omega)}{\sin\omega}\tan(\omega - \delta)\frac{f}{m},$$

$$\beta^2 = -\frac{2(b - 2c\cos\omega)}{\sin\omega}\tan\delta\,\frac{f}{m}.$$

14. To find whether $\mu = 0$ or ∞, when $b - 2c\cos\omega = 0$.

$$\mu = \frac{a' - 2c\sin^2\omega - \sqrt{a'^2 + m'}}{b'\sin\omega};$$

$$a' = a + c - b\cos\omega, \quad b' = b - 2c\cos\omega, \quad \text{or } b = b' + 2c\cos\omega;$$

$$\therefore\ a' = a + c - b'\cos\omega - 2c\cos^2\omega = a - c\cos 2\omega - b'\cos\omega,$$

$$\text{and } a' - 2c\sin^2\omega = a - c - b'\cos\omega:$$

also $a'^2 + m' = a^2 - 2ac\cos 2\omega + c^2\cos^2 2\omega - 2b'\cos\omega\,(a - c\cos 2\omega)$

$$+ b'^2\cos^2\omega + (b'^2 + 4b'c\cos\omega + 4c^2\cos^2\omega)\sin^2\omega - 4ac\sin^2\omega$$

$$= a^2 - 2ac + c^2 - 2b'\cos\omega\,(a - c\cos 2\omega - 2c\sin^2\omega) + b'^2$$

$$= (a-c)^2 - 2b'\cos\omega\,(a-c) + b'^2 = (a - c - b'\cos\omega)^2 + b'^2\sin^2\omega.$$

Now if $a > c$;

$$\mu = \frac{(a - c - b'\cos\omega) - \sqrt{(a - c - b'\cos\omega)^2 + b'^2\sin^2\omega}}{b'\sin\omega}$$

$$= -\frac{b'\sin\omega}{2(a - c - b'\cos\omega)} = 0, \text{ when } b' = 0;$$

and when $a < c$,

$$\mu = \frac{a - c - b' \cos \omega + (a - c - b' \cos \omega)}{b' \sin \omega} = \infty :$$

hence the axis-major is parallel to the axes of x or at right angles to it according as $a >$ or $< c$.

When $a > c$

$$A^2 = -(a' + \sqrt{a'^2 + m'}) \frac{hk - g}{b}$$

$$= -\{a - c \cos 2\omega + (a - c)\} \frac{hk - g}{b} = -2(a - c \cos^2 \omega) \frac{hk - g}{b},$$

$$B^2 = -(a' - \sqrt{a'^2 + m'}) \frac{hk - g}{b}$$

$$= -\{a - c \cos 2\omega - (a - c)\} \frac{hk - g}{b} = -2c \sin^2 \omega \left(\frac{hk - g}{b}\right).$$

When $a < c$,

$$A^2 = -\{a - c \cos 2\omega + (c - a)\} \frac{hk - g}{b} = -2c \sin^2 \omega \frac{hk - g}{b};$$

$$B^2 = -\{a - c \cos^2 \omega - (a - c)\} \frac{hk - g}{b} = -2(a - c \cos^2 \omega) \frac{hk - g}{b}.$$

(α) When $a = c$, $\mu - \frac{1}{\mu} = -2 \cot \omega = -2 \cot 2\delta$; $\therefore \delta = \frac{\omega}{2}$.

(β) When $b^2 < 4ac$ and $hk - g = 0$, $A = B = 0$; which is the condition that the equation of the second order may be a point.

(γ) When b^2 is not less than $4ac$, and $hk - g = 0$, the equation represents two straight lines.

(δ) To determine which sign ought to be used in the expression for $\mu + \frac{1}{\mu}$.

We have $(\beta - k)^2 = \frac{(b - 2c \cos \omega) \tan \delta}{\sin \omega} \left(\frac{hk - g}{b}\right)$

$$= \frac{(b - 2c \cos \omega) \tan \delta}{\sin \omega} \left(\frac{mM - H^2}{2am^2}\right),$$

$$\text{and } y = -\frac{bx+d}{2a} \pm \frac{\sqrt{m(x+h)^2 + \dfrac{mM-H^2}{m}}}{2a};$$

therefore when m is negative, $mM - H^2$ must be negative, otherwise every value of x would make y impossible; hence $(b - 2c\cos\omega)\tan\delta$ is negative, since a is supposed positive; and

$$(b - 2c\cos\omega)\left(\mu + \frac{1}{\mu}\right) = (b - 2c\cos\omega)\,\mu\,.\left(\frac{\mu^2+1}{\mu^2}\right)$$

is negative; but

$$(b - 2c\cos\omega)\left(\mu + \frac{1}{\mu}\right) = \pm\frac{2\sqrt{a'^2 + m'}}{\sin\omega};$$

$\therefore$ when m is negative, the negative sign only of $\sqrt{a'^2 + m'}$ is admissible.

When m is positive, and a positive, the positive or negative sign must be used according as $mM - H^2$ is positive or negative; but this is the condition that the curve may or may not be continuous for all values of x; for if $mM > H^2$, every value of x will make the quantity under the radical in the value of y positive, or y is possible for every value of x, and the curve is unlimited from $x = +\infty$ to $x = -\infty$.

If $mM < H^2$, the quantity under the radical

$$= \sqrt{m\left\{(x+h)^2 - p^2\right\}},$$

and for all values of x between $p - h$, and $-p - h$, y is impossible; or the curve does not extend indefinitely on both sides of the axis of y.

15. (1) If $ay^2 + bxy + cx^2 + dy + ex + f = \phi(x, y) = 0$ be the equation to a curve of the second degree; and $b^2 - 4ac = 0$, in which case the curve represents a parabola; to transform the origin to the vertex, the axes being rectangular.

Transform the origin to a point α_1, β_1 in the curve by making $x = x' + \alpha_1$, and $y = y' + \beta_1$;

$$\therefore\ ay'^2 + bx'y' + cx'^2 + d'y' + e'x' = 0;$$

where $\phi(\alpha_1, \beta_1) = 0$, $d' = 2a\beta_1 + b\alpha_1 + d$, and $e' = b\beta_1 + 2c\alpha_1 + e$.

Next let the equation be transformed to polar co-ordinates by putting $x' = \rho\cos\theta$, $y' = \rho\sin\theta$;

$$\therefore\ (a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta)\,\rho + (d'\sin\theta + e'\cos\theta) = 0,$$

$$\text{or}\ (b\sin\theta + 2c\cos\theta)^2\rho + 4c\,(d'\sin\theta + e'\cos\theta) = 0;$$

and if $\dfrac{d'}{e'} = -\dfrac{2c}{b} = \tan\delta$; we have

$$b^2\sec^2\delta\sin^2(\theta - \delta)\,\rho + 4ce'\sec\delta\cos(\theta - \delta) = 0,$$

$$\text{or}\ \sin^2(\theta - \delta)\,\rho + \frac{e'\cos\delta}{a}\cos(\theta - \delta) = 0. \quad \ldots\ldots\ (1)$$

Now the polar equation to a parabola, whose latus rectum is $2L$, having its axis inclined at an angle δ to the axis of x, referred to the vertex as the origin, is

$$\rho^2\sin^2(\theta - \delta) = 2L\rho\cos(\theta - \delta),$$

$$\text{or}\ \rho\sin^2(\theta - \delta) = 2L\cos(\theta - \delta);$$

hence equation (1) is the equation to a parabola from the vertex, whose axis makes an angle (δ) with the axis of x, and latus rectum $2L = -\dfrac{e'\cos\delta}{a}$. And $\phi(x, y) = 0$ is the equation to a parabola, the co-ordinates of whose vertex are α_1, β_1; having its axis inclined at an angle δ to the axis of x, and latus rectum $= -\dfrac{e'\cos\delta}{a}$.

(2) To find α_1, β_1, L.

We have $\dfrac{d'}{e'} = \dfrac{2a\beta_1 + b\alpha_1 + d}{b\beta_1 + 2c\alpha_1 + e} = -\dfrac{2c}{b}$;

$$\therefore\ 2(a + c)(b\beta_1 + 2c\alpha_1) = -(bd + 2ce),$$

$$\text{or}\ 2(a + c)(b\beta_1 + 2c\alpha_1 + e) = 2ae - bd = H,$$

$$\text{and}\ b\beta_1 + 2c\alpha_1 + e = e' = \frac{H}{2(a + c)};$$

$$\text{also } 4c\phi(\alpha_1, \beta_1) = 0,$$

$$\text{and } e'^2 - 4c\phi(\alpha_1, \beta_1) = -2K\beta_1 + (e^2 - 4cf);$$

$$\text{or } e'^2 = \frac{H^2}{4(a+c)^2} = -2K\beta_1 + N;$$

$$\text{and } \beta_1 = \frac{N}{2K} - \frac{H^2}{8K(a+c)^2} = \frac{N}{2K} - \frac{aK}{8c(a+c)^2}. \quad \ldots\ldots(2)$$

Similarly, $2(a+c)d' = K$, and

$$d'^2 - 4a\phi(\alpha_1, \beta_1) = -2H\alpha_1 + M;$$

$$\therefore\ \alpha_1 = \frac{M}{2H} - \frac{cH}{8a(a+c)^2}. \quad \ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{And } L = -\frac{e'\cos\delta}{2a} = -\frac{H\cos\delta}{4a(a+c)};$$

and $\cos\delta$ and $\tan\delta$ may be considered to have the same sign;

$$\therefore\ \sec\delta = -\frac{\sqrt{4c^2 + 4ac}}{b};$$

$$\text{or } L = \frac{b(2ae - bd)}{8a\sqrt{c}(a+c)^{\frac{3}{2}}} = \frac{eb - 2cd}{4\sqrt{c}(a+c)^{\frac{3}{2}}} = -\frac{K}{4\sqrt{c}(a+c)^{\frac{3}{2}}}. \quad (4)$$

16. (1) When the axes are oblique, let the origin be transferred to a point α_1, β_1 as before, and the equation then reduced to polar co-ordinates;

$$\therefore\ \{a\sin^2\theta + b\sin\theta\sin(\omega-\theta) + c\sin^2(\omega-\theta)\}\rho$$
$$+ \sin\omega\,\{d'\sin\theta + e'\sin(\omega-\theta)\} = 0;$$

$$\text{or } \{(a - b\cos\omega + c\cos^2\omega)\sin^2\theta + \sin\omega\,(b - 2c\cos\omega)\sin\theta\cos\theta + c\sin^2\omega\cos^2\theta\}\rho$$
$$+ \sin\omega\,\{(d' - e'\cos\omega)\sin\theta + e'\sin\omega\cos\theta\} = 0;$$

$$\text{hence } \{2c\sin\omega\cos\theta + (b - 2c\cos\omega)\sin\theta\}^2\rho$$
$$+ 4c\sin\omega\,\{(d' - e'\cos\omega)\sin\theta + e'\sin\omega\cos\theta\} = 0;$$

$$\text{let } \frac{d' - e'\cos\omega}{e'\sin\omega} = -\frac{2c\sin\omega}{b - 2c\cos\omega} = \tan\delta\,; \;\ldots\ldots\; (1)$$

$$\therefore\; \frac{4c^2\sin^2\omega}{\sin^2\delta}\sin^2(\theta-\delta)\,\rho + 4ce'\frac{\sin^2\omega}{\cos\delta}\cos(\theta-\delta) = 0,$$

$$\text{and } \sin^2(\theta-\delta)\,\rho + \frac{e'\sin^2\delta}{c\cos\delta}\cos(\theta-\delta) = 0\,;$$

which is the equation to a parabola whose axis makes an angle δ with the axis of x, and latus rectum

$$2L = -\frac{e'\sin^2\delta}{c\cos\delta}.$$

(2) To find α_1, β_1 and L.

$$\text{We have } \frac{d'}{e'} = \cos\omega - \frac{2c\sin^2\omega}{b - 2c\cos\omega} = \frac{b\cos\omega - 2c}{b - 2c\cos\omega}\,;$$

$$\text{or } \frac{2a\beta_1 + b\alpha_1 + d}{b\beta_1 + 2c\alpha_1 + e} = \frac{b\cos\omega - 2c}{b - 2c\cos\omega}\,;$$

$$\text{hence } \frac{2a}{b} - \frac{2ae - bd}{b(b\beta_1 + 2c\alpha_1 + e)} = \frac{b\cos\omega - 2c}{b - 2c\cos\omega}\,;$$

$$\text{and } \frac{H}{be'} = \frac{2a}{b} - \frac{b\cos\omega - 2c}{b - 2c\cos\omega}$$

$$= \frac{2ab - 2b^2\cos\omega + 2bc}{b(b - 2c\cos\omega)} = \frac{2(a - b\cos\omega + c)}{b - 2c\cos\omega} = \frac{2a'}{b'}\,;$$

$$\text{and } e' = \frac{Hb'}{2ba'}\,; \;\text{ or } b\beta_1 + 2c\alpha_1 + e = \frac{Hb'}{2ba'} \;\ldots\ldots\; (2)$$

is the equation to the axis;

$$\text{also } L = -\frac{e'}{2c}\tan\delta\sin\delta = \frac{H\sin\omega}{2ba'}\sin\delta = \frac{H\sin\omega}{2ba'}\frac{\sqrt{c}\sin\omega}{\sqrt{a'}}\,;$$

since $\sin\delta$ may always be considered positive;

$$\therefore\; L = \frac{H\sin^2\omega\,.\sqrt{c}}{2ba'^{\frac{3}{2}}}\,;$$

hence $L = \dfrac{Hc\sin^2\omega}{2b\sqrt{c}\,a'^{\frac{3}{2}}} = \dfrac{(2ae-bd)\,c\sin^2\omega}{2b\sqrt{c}\,a'^{\frac{3}{2}}} = \dfrac{(be-2cd)\sin^2\omega}{4\sqrt{c}\,.\,a'^{\frac{3}{2}}}$,

$$\text{or } L = \frac{-K\sin^2\omega}{4\sqrt{c}\,.\,a'^{\frac{3}{2}}} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3).$$

As in the last case $e'^2 = -2K\beta_1 + N$,

$$\therefore\; \beta_1 = \frac{N}{2K} - \frac{e'^2}{2K} = \frac{N}{2K} - \frac{H^2 b'^2}{8Kb^2a'^2},$$

$$\text{and } \frac{H}{K} = -\frac{b}{2c};\;\therefore\; H^2 = \frac{b^2}{4c^2}K^2,$$

$$\text{or } \beta_1 = \frac{N}{2K} - \frac{Kb'^2}{32c^2a'^2} = \frac{N}{2K} - \frac{K}{8a'^2c}(a - b\cos\omega + c\cos^2\omega)$$

$$= \frac{N}{2K} - \frac{K}{8a'^2c}(a' - c\sin^2\omega) = \frac{N}{2K} - \frac{K}{8a'c} + \frac{K\sin^2\omega}{8a'^2} \ldots (4)$$

$$\text{similarly, } \alpha_1 = \frac{M}{2H} - \frac{H}{8a'a} + \frac{H\sin^2\omega}{8a'^2} \ldots (5).$$

17. Let $b^2 < 4ac$, and the co-ordinates rectangular, to find the polar equation from the centre.

Change the origin to the centre by putting $x = x' + h$, $y = y' + k$;

$$\therefore\; ay'^2 + bx'y' + cx'^2 + \phi(h, k) = 0.$$

Again let this equation be transformed to polar co-ordinates by putting $x' = \rho\cos\theta$, $y' = \rho\sin\theta$;

$$\therefore\; (a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta)\,\rho^2 + \phi(h, k) = 0;\;\ldots (1)$$

$$\text{or } \left[a - \{(a-c)\cos^2\theta - b\sin\theta\cos\theta\}\right]\rho^2 + \phi(h, k) = 0;$$

and if $\tan 2\delta = -\dfrac{b}{a-c}$, we have

$$\{a - (a-c)\sec 2\delta\cos\theta.\cos(\theta - 2\delta)\}\,\rho^2 + \phi(h, k) = 0;\;\ldots (2)$$

but the polar equation to the ellipse, referred to the centre, whose axis-major makes an $\angle\,\delta$ with the axis of x is

$$\{1 - E^2\sin^2\delta - E^2\cos\theta\cos(\theta - 2\delta)\}\,\rho^2 = B^2;$$

hence equation (1) may be made to coincide with this equation by putting

$$\frac{1-E^2\sin^2\delta}{B^2} = \frac{-a}{\phi(h,k)} \text{ and } \frac{E^2}{B^2} = -\frac{(a-c)\sec 2\delta}{\phi(h,k)}; \ldots (3)$$

$$\therefore \frac{2-2E^2\sin^2\delta}{E^2-2E^2\sin^2\delta} = \frac{2a}{a-c}; \text{ or } \frac{2-E^2}{E^2\cos 2\delta} = \frac{a+c}{a-c};$$

$$\text{and } \frac{2-E^2}{E^2} = \frac{a+c}{(a-c)\sec 2\delta} = \frac{a+c}{\sqrt{(a+c)^2+m}} = \frac{a'}{\sqrt{a'^2+m}},$$

the negative sign being inadmissible since

$$\frac{2}{E^2} = \left(1 \pm \frac{a'}{\sqrt{a'^2+m}}\right), \text{ and } \frac{a'}{\sqrt{a'^2+m}} > 1;$$

$$\therefore 1 - \frac{a'}{\sqrt{a'^2+m}} \text{ is negative and cannot } = \frac{2}{E^2}.$$

$$\text{hence } \frac{2}{E^2} = 1 + \frac{a'}{\sqrt{a'^2+m}},$$

$$\text{or } E^2 = \frac{2\sqrt{a'^2+m}}{a'+\sqrt{a'^2+m}} = 1 + \frac{(a'-\sqrt{a'^2+m})^2}{m}; \ldots (4)$$

$$\text{also } B^2 = -\frac{E^2\phi(h,k)}{(a-c)\sec 2\delta} = -\frac{E^2\phi(h,k)}{\sqrt{a'^2+m}} = -\frac{2\phi(h,k)}{(a'+\sqrt{a'^2+m})};$$

$$\text{hence } B^2 = \frac{2\phi(h,k)}{m}(a'-\sqrt{a'^2+m}) \ldots\ldots\ldots\ldots (5)$$

$$A^2 = \frac{B^2}{1-E^2} = \frac{2\phi(h,k)}{m}(a'+\sqrt{a'^2+m}) \ldots\ldots\ldots (6)$$

$$L^2 = B^2(1-E^2) = -2\frac{\phi(h,k)}{m^2}(a'-\sqrt{a'^2+m})^3 \ldots (7).$$

When $b^2 < 4ac$, $a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$

$$= \frac{1}{4a}\left\{(2a\sin\theta + b\cos\theta)^2 + (4ac-b^2)\cos^2\theta\right\}$$

is positive; or from equation (1), $\phi(h,k)$ is negative.

From Art. 9, if $e' = b\beta + 2c\alpha + e$; and $\phi(\alpha, \beta) = \Phi$; whatever be α, β we have

$$e'^2 - 4c\Phi = m\left\{(\beta - k)^2 + \frac{2c}{b}(hk - g)\right\},$$

and when $\alpha = h$, $\beta = k$, e' becomes $= 0$; and $\Phi = \phi(h, k)$;

$$\therefore\ -4c\phi(h, k) = 2c\frac{m}{b}(hk - g), \text{ or } 2\frac{\phi(h, k)}{m} = -\left(\frac{hk - g}{b}\right).$$

Hence $A^2 = -(a' + \sqrt{a'^2 + m})\dfrac{hk - g}{b}$; .... (8)

$$B^2 = -(a' - \sqrt{a'^2 + m})\left(\frac{hk - g}{b}\right); \text{ .... } (9)$$

$$L^2 = \frac{(a' - \sqrt{a'^2 + m})^3}{m}\left(\frac{hk - g}{b}\right) \text{ .... } (10)$$

In the ellipse $\phi(h, k)$ is negative, and since $\dfrac{E^2}{B^2}$ is positive, equation (3) shews that in the ellipse $(a - c)\sec 2\delta$ is positive;

$$\therefore\ (a - c)\sec 2\delta = +\sqrt{a'^2 + m}.$$

In the hyperbola equation (3) becomes

$$\frac{E^2}{-B^2} = -\frac{(a - c)\sec 2\delta}{\phi(h, k)}, \text{ or } \frac{E^2}{B^2} = \frac{(a - c)\sec 2\delta}{\phi(h, k)};$$

and since in this case m is positive,

$$\phi(h, k) = -\frac{m}{2}\left(\frac{hk - g}{b}\right) = -\frac{M - mh^2}{4a} = -\frac{Mm - H^2}{4am},$$

which when a is positive, will be negative or positive according as the curve extends indefinitely on both sides of the axis of y or not.

Also $$\frac{1 - E^2\sin^2\delta}{B^2} = \frac{a}{\phi(h, k)};$$

$$\therefore\ \frac{2}{E^2} - 1 = \frac{a + c}{(a - c)\sec 2\delta};$$

and if $\phi(h, k)$ be positive, $(a-c)\sec 2\delta$ is positive $=\sqrt{a'^2+m}$;

$$\therefore \frac{2}{E^2}=1+\frac{a'}{\sqrt{a'^2+m}}, \text{ or } E^2=\frac{2\sqrt{a'^2+m}}{a'+\sqrt{a'^2+m}}; \ldots (11)$$

$$B^2=\frac{E^2.\phi(h, k)}{(a-c)\sec 2\delta}=2\frac{(\sqrt{a'^2+m}-a')}{m}\phi(h, k); \quad (12)$$

$$A^2=\frac{B^2}{E^2-1} \qquad =2\left(\frac{\sqrt{a'^2+m}+a'}{m}\right)\phi(h, k); \quad (13)$$

$$\text{or } B^2=-(\sqrt{a'^2+m}-a')\left(\frac{hk-g}{b}\right); \ \ldots\ldots\ldots (14)$$

$$A^2=-(\sqrt{a'^2+m}+a')\left(\frac{hk-g}{b}\right); \ \ldots\ldots\ldots (15)$$

If $\phi(h, k)$ be negative, $(a-c)\sec 2\delta$ is negative $=-\sqrt{a'^2+m}$;

$$\therefore \frac{2}{E^2}=1-\frac{a'}{\sqrt{a'^2+m}}, \text{ and } E^2=\frac{2\sqrt{a'^2+m}}{\sqrt{a'^2+m}-a'};$$

$$B^2=\frac{E^2\phi(h, k)}{(a-c)\sec 2\delta}=-\frac{2\phi(h, k)}{\sqrt{a'^2+m}-a'}$$

$$=-\frac{2(a'+\sqrt{a'^2+m})}{m}\phi(h, k); \quad (16)$$

$$A^2=\frac{B^2}{E^2-1} \qquad =-\frac{2(\sqrt{a'^2+m}-a')}{m}\phi(h, k); \quad (17)$$

$$\text{or } B^2=(a'+\sqrt{a'^2+m})\left(\frac{hk-g}{b}\right); \ \ldots\ldots\ldots (18)$$

$$A^2=(\sqrt{a'^2+m}-a')\left(\frac{hk-g}{b}\right). \ \ldots\ldots\ldots (19)$$

18. Let $b^2<4ac$, and the co-ordinates inclined at an angle ω, to find the polar equation from the centre.

Change the origin as before to the centre, and transform the equation to polar co-ordinates, and the resulting equation becomes

$$\{(a - b\cos\omega + c\cos^2\omega)\sin^2\theta + b'\sin\omega\sin\theta\cos\theta + c\sin^2\omega\cos^2\theta\}\rho^2$$
$$+ \sin^2\omega\,\phi(h, k) = 0; \text{ or}$$

$$\{(a - b\cos\omega + c\cos^2\omega) + b'\sin\omega\sin\theta\cos\theta - (a - b\cos\omega + c\cos 2\omega)\cos^2\theta\}\rho^2$$
$$+ \sin^2\omega\,.\,\phi(h, k) = 0;$$

hence $\{(a' - c\sin^2\omega) + b'\sin\omega\,.\,\sin\theta\cos\theta - (a' - 2c\sin^2\omega)\cos^2\theta\}\rho^2$

$$+ \sin^2\omega\,.\,\phi(h, k) = 0;$$

$$\therefore\ \{(a' - c\sin^2\omega) - (a' - 2c\sin^2\omega)\sec 2\delta\cos\theta\cos(\theta - 2\delta)\}\rho^2$$
$$+ \sin^2\omega\,\phi(h, k) = 0; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

by putting $\tan 2\delta = \dfrac{-b'\sin\omega}{a' - 2c\sin^2\omega}$; $\ldots\ldots\ldots\ldots\ldots\ldots$ (2)

and the polar equation to an ellipse from the centre, whose axis-major makes an angle (δ) with the axis of x, is

$$\{(1 - E^2\sin^2\delta) - E^2\cos\theta\cos(\theta - 2\delta)\}\rho^2 = B^2;$$

and equation (1) will coincide with this equation if

$$\frac{1 - E^2\sin^2\delta}{B^2} = -\frac{(a' - c\sin^2\omega)}{\sin^2\omega\,\phi(h, k)};$$

$$\text{and } \frac{E^2}{B^2} = -\frac{(a' - 2c\sin^2\omega)\sec 2\delta}{\sin^2\omega\,\phi(h, k)}; \ldots\ldots\ldots\ldots (3)$$

$$\text{or } \frac{2 - 2E^2\sin^2\delta}{E^2 - 2E^2\sin^2\delta} = \frac{2a' - 2c\sin^2\omega}{a' - 2c\sin^2\omega};$$

$$\text{hence } \frac{2 - E^2}{E^2\cos 2\delta} = \frac{a'}{a' - 2c\sin^2\omega};$$

$$\text{or } \frac{2}{E^2} = 1 + \frac{a'}{(a' - 2c\sin^2\omega)\sec 2\delta}.$$

Now in the ellipse $\frac{E^2}{B^2}$ is positive, and $\phi(h, k)$ is negative; $\therefore$ $(a' - 2c\sin^2\omega)\sec 2\delta$ is positive from equation (3); hence equation (2) gives

$$(a' - 2c\sin^2\omega)\sec 2\delta = \sqrt{(a' - 2c\sin^2\omega)^2 + b'^2\sin^2\omega} = \sqrt{a'^2 + m'};$$

$$\text{or } \frac{2}{E^2} = 1 + \frac{a'}{\sqrt{a'^2 + m'}};$$

$$\therefore\ E^2 = \frac{2\sqrt{a'^2 + m'}}{a' + \sqrt{a'^2 + m'}}; \quad \ldots\ldots\ldots\ldots\ldots\ (4)$$

$$B^2 = \frac{-E^2\sin^2\omega \cdot \phi(h, k)}{(a' - 2c\sin^2\omega)\sec 2\delta} = \frac{2\sin^2\omega}{m'}(a' - \sqrt{a'^2 + m'})\,\phi(h, k)$$

$$= 2(a' - \sqrt{a'^2 + m'})\frac{\phi(h, k)}{m}; \quad (5)$$

$$A^2 = \frac{B^2}{1 - E^2} \qquad = 2(a' + \sqrt{a'^2 + m'})\frac{\phi(h, k)}{m}; \quad (6)$$

$$L^2 = B^2(1 - E^2) \qquad = \frac{-2(a' - \sqrt{a'^2 + m'})^3}{m'}\frac{\phi(h, k)}{m}; \quad (7)$$

and as before, $\dfrac{2\phi(h, k)}{m} = -\dfrac{(hk - g)}{b};$

$$\therefore\ A^2 = -(a' + \sqrt{a'^2 + m'})\left(\frac{hk - g}{b}\right); \quad \ldots\ldots\ldots\ldots\ (8)$$

$$B^2 = -(a' - \sqrt{a'^2 + m'})\left(\frac{hk - g}{b}\right); \quad \ldots\ldots\ldots\ldots\ (9)$$

$$L^2 = \frac{(a' - \sqrt{a'^2 + m'})^3}{m'}\left(\frac{hk - g}{b}\right). \ldots\ldots\ldots\ldots\ (10)$$

In the hyperbola, equation (3) becomes

$$\frac{1 - E^2\sin^2\delta}{B^2} = \frac{a' - c\sin^2\omega}{\sin^2\omega\,\phi(h, k)}, \text{ and } \frac{E^2}{B^2} = \frac{(a' - 2c\sin^2\omega)\sec 2\delta}{\sin^2\omega\,\phi(h, k)}.$$

Hence, (α) if $\phi(h, k)$ be positive $(a' - 2c\sin^2\omega)\sec 2\delta$ is positive;

$$\text{and } \frac{2}{E^2} = 1 + \frac{a'}{(a' - 2c\sin^2\omega)\sec 2\delta} = 1 + \frac{a'}{\sqrt{a'^2 + m'}},$$

$$B^2 = \frac{E^2 \sin^2\omega \, . \, \phi(h, k)}{(a' - 2c\sin^2\omega)\sec 2\delta} = 2\left(\sqrt{a'^2 + m'} - a'\right)\frac{\phi(h, k)}{m}. \quad (11)$$

$$A^2 = \frac{B^2}{E^2 - 1} = 2\left(\sqrt{a'^2 + m'} + a'\right)\frac{\phi(h, k)}{m}, \quad \ldots\ldots\ldots\ldots \quad (12)$$

$$\text{or } A^2 = -\left(a' + \sqrt{a'^2 + m'}\right)\left(\frac{hk - g}{b}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$B^2 = \left(a' - \sqrt{a'^2 + m'}\right)\frac{hk - g}{b}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

(β) If $\phi(h, k)$ be negative,

$$(a' - 2c\sin^2\omega)\sec 2\delta = -\sqrt{a'^2 + m'},$$

$$\text{and } \frac{2}{E^2} = 1 - \frac{a'}{\sqrt{a'^2 + m'}}, \text{ and } E^2 = \frac{2\sqrt{a'^2 + m'}}{\sqrt{a'^2 + m'} - a'};$$

$$B^2 = -2\left(a' + \sqrt{a'^2 + m'}\right)\frac{\phi(h, k)}{m} = \left(a' + \sqrt{a'^2 + m'}\right)\left(\frac{hk - g}{b}\right), \quad (15)$$

$$A^2 = \frac{B^2}{E^2 - 1} = -2\left(\sqrt{a'^2 + m'} - a'\right)\frac{\phi(h, k)}{m} = \left(\sqrt{a'^2 + m'} - a'\right)\left(\frac{hk - g}{b}\right). \quad (16)$$

$$\text{Also since } \tan 2\delta = -\frac{b'\sin\omega}{a' - 2c\sin^2\omega},$$

we have,

(1) If $b' = 0$, $\tan 2\delta = 0$; $\therefore$ $\delta = 0$, or 90.

(2) If $a = c$, $\tan 2\delta = -\dfrac{b'\sin\omega}{a' - 2c\sin^2\omega}$

$$= -\frac{(b - 2c\cos\omega)\sin\omega}{2c\cos^2\omega - b\cos\omega} = \tan\omega, \text{ or } \delta = \frac{\omega}{2}.$$

(3) If $a' - 2c\sin^2\omega = 0$, or $a - b\cos\omega + c\cos 2\omega = 0$,

$$\tan 2\delta = \infty\text{, and } \delta = 45.$$

(4) If $a' - c\sin^2\omega = 0$, or $a - b\cos\omega + c\cos^2\omega = 0$,

$$\frac{1 - E^2\sin^2\delta}{B^2} = 0\,;\ \therefore\ E = \operatorname{cosec}\delta.$$

(5) To deduce the latus rectum, and co-ordinates of the vertex of the parabola by making m vanish.

The co-ordinates of the extremities of the axis-major, are

$$\alpha_1 = h - \sqrt{\frac{A^2\sin^2(\omega - \delta)}{\sin^2\omega}}\,;\quad \beta_1 = k - \sqrt{\frac{A^2\sin^2\delta}{\sin^2\omega}}:$$

$$\text{also } \frac{hk - g}{b} = -\frac{H^2 - mM}{2am^2} = -\frac{K^2 - mN}{2cm^2}\,;$$

and when m' becomes very small,

$$\frac{(a' - \sqrt{a'^2 + m'})^3}{m'} = -\frac{m'^2}{8a'^3}\text{, nearly;}$$

$$\therefore\ L^2 = -\frac{m'^2}{8a'^3} \times -\frac{K^2 - mN}{2cm^2} = \frac{(K^2 - mN)\sin^4\omega}{16ca'^3},$$

$$\text{hence } L = \pm\frac{K\sin^2\omega}{4\sqrt{ca'^3}}\left(1 - \frac{mN}{2K^2}\right)\,;\ \ldots\ldots\ldots\ldots\ldots \text{(A)}$$

$$\text{and when } m \text{ vanishes } L = \pm\frac{K\sin^2\omega}{4\sqrt{ca'^3}}.\ \ldots\ldots\ldots\ldots\ldots \text{(B)}$$

$$\text{Also } \tan\delta = \frac{a - b\cos\omega + c\cos 2\omega - \sqrt{a'^2 + m'}}{b'\sin\omega},$$

$$\text{or } \mu = \frac{a' - 2c\sin^2\omega - \left(a' + \dfrac{m'}{2a'}\right)}{b'\sin\omega} = -\frac{2c\sin\omega}{b'}\left(1 + \frac{m}{4a'c}\right);$$

$$\operatorname{cosec} 2\delta = -\frac{\sqrt{a'^2 + m'}}{b' \sin\omega} = -\frac{\left(a' + \dfrac{m'}{2a'}\right)}{b' \sin\omega};$$

$$\sin^2\delta = \frac{\tan\delta}{2 \operatorname{cosec} 2\delta} = -\frac{c \sin\omega}{b'}\left(1 + \frac{m}{4a'c}\right) \times \frac{-b' \sin\omega}{a' + \dfrac{m'}{2a'}};$$

$$\text{or } \sin^2\delta = \frac{c \sin^2\omega}{a'}\left(1 + \frac{m}{4a'c} - \frac{m'}{2a'^2}\right)$$

$$= \frac{c \sin^2\omega}{a'}\left\{1 + \frac{m}{4a'^2c}(a - b\cos\omega + c\cos 2\omega)\right\};$$

$$\text{hence } \beta_1 = \frac{K}{m} - \sqrt{\left(2a' + \frac{m'}{2a'}\right)\left(\frac{K^2 - mN}{2cm^2}\right)\frac{c}{a'}\left\{1 + \frac{m}{4a'^2c}(a - b\cos\omega + c\cos 2\omega)\right\}}$$

$$= \frac{K}{m} - \sqrt{\frac{K^2}{m^2}\left\{1 - \frac{mN}{K^2} + \frac{m'}{4a'^2} + \frac{m}{4a'^2c}(a - b\cos\omega + c\cos 2\omega)\right\}}$$

$$= \frac{K}{m} - \frac{K}{m}\left\{1 - \frac{mN}{2K^2} + \frac{m'}{8a'^2} + \frac{m}{8a'^2c}(a - b\cos\omega + c\cos 2\omega)\right\},$$

$$\text{or } \beta_1 = \frac{N}{2K} - \frac{K}{8a'^2c}(a - b\cos\omega + c\cos^2\omega). \;\ldots\ldots\; \text{(C)}$$

$$\text{Similarly, } \alpha_1 = \frac{M}{2H} - \frac{H}{8a'^2a}(c - b\cos\omega + a\cos^2\omega). \;\ldots\; \text{(D)}$$

When the two axes are tangents to the curve, $M = 0$, $N = 0$; and $\dfrac{hk - g}{b} = -\left(\dfrac{mh^2 - M}{2am}\right) = -\left(\dfrac{mk^2 - N}{2cm}\right)$;

$$\therefore \frac{hk - g}{b} = -\frac{h^2}{2a} = -\frac{k^2}{2c},$$

either of which expressions may be put for $\dfrac{hk - g}{b}$ in the values of A, B, L.

19. To determine the position of the axis, the co-ordi-

nates of the vertex and the latus rectum of the parabola whose equation is

$$ay^2 - bxy + cx^2 - dy - ex + f = \phi(x, y) = 0;$$

$$\text{where } b^2 = 4ac; \; d^2 = 4af, \text{ and } e^2 = 4cf;$$

the co-ordinate axes being inclined to each other at a given angle ω.

Change the origin to a point α, β in the curve; and transform the equation to polar co-ordinates;

$$\therefore \; \{a \sin^2\theta - b \sin\theta \sin(\omega - \theta) + c \sin^2(\omega - \theta)\} \rho$$
$$+ \sin\omega \{d' \sin\theta + e' \sin(\omega - \theta)\} = 0;$$

$$\text{where } d' = 2a\beta - b\alpha - d; \quad e' = -b\beta + 2c\alpha - e;$$

$$\text{and } \phi(\alpha, \beta) = 0;$$

or $\{(a + b\cos\omega + c\cos^2\omega)\sin^2\theta - \sin\omega(b + 2c\cos\omega)\sin\theta\cos\theta$

$+ c\sin^2\omega\cos^2\theta\}\rho + \sin\omega\{(d - e'\cos\omega)\sin\theta + e'\sin\omega\cos\theta\} = 0;$

and since $b^2 = 4ac$, $\{(b + 2c\cos\omega)\sin\theta - 2c\sin\omega\cos\theta\}^2 \rho$

$$+ 4c\sin\omega\{(d' - e'\cos\omega)\sin\theta + e'\sin\omega\cos\theta\} = 0.$$

$$\text{Let} \quad \frac{d' - e'\cos\omega}{e'\sin\omega} = \frac{2c\sin\omega}{b + 2c\cos\omega} = \tan\delta;$$

$$\therefore \; \frac{4c^2\sin^2\omega}{\sin^2\delta}\sin^2(\theta - \delta)\rho + \frac{4ce'\sin^2\omega}{\cos\delta} \cdot \cos(\theta - \delta) = 0;$$

which is the equation to a parabola whose axis makes an angle δ with the axis of x, and latus rectum

$$2L = -\frac{e'\sin^2\delta}{c\cos\delta} = -\frac{2e'\sin\omega}{b + 2c\cos\omega}\sin\delta.$$

$$\text{Also } \frac{d'}{e'} = \frac{2a\beta - b\alpha - d}{2c\alpha - b\beta - e} = \cos\omega + \frac{2c\sin^2\omega}{b + 2c\cos\omega} = \frac{b\cos\omega + 2c}{b + 2c\cos\omega};$$

therefore by reduction

$$2(a + b\cos\omega + c)(b\beta - 2c\alpha + e) = db + 2ae + (2cd + be)\cos\omega;$$

$$\text{but } db = 2ae, \text{ and } 2cd = be;$$

$$\therefore \; -2(a + b\cos\omega + c)e' = 2d(b + 2c\cos\omega);$$

$$\text{hence } 2L = \frac{2d\sin\omega\sin\delta}{a + b\cos\omega + c},$$

$$\text{and } \sin\delta = \frac{\sqrt{c}\,.\sin\omega}{\sqrt{a + b\cos\omega + c}};$$

$$\therefore \; L = \frac{d\sqrt{c}\sin^2\omega}{(a + b\cos\omega + c)^{\frac{3}{2}}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

The axes of x and y are both tangents to the parabola; and if a_1, b_1 be the distances from the origin at which the curve meets the axes,

$$a_1 = \sqrt{\frac{f}{c}}, \quad b_1 = \frac{d}{2a} = \sqrt{\frac{f}{a}};$$

$$\text{or } \frac{d}{a} = 2b_1, \quad \sqrt{\frac{c}{a}} = \frac{b_1}{a_1}, \quad \frac{b}{a} = \frac{2b_1}{a_1};$$

$$\therefore \; L = \frac{\frac{d}{a}\sqrt{\frac{c}{a}}\sin^2\omega}{\left(1 + \frac{b}{a}\cos\omega + \frac{c}{a}\right)^{\frac{3}{2}}} = \frac{2b_1\frac{b_1}{a_1}\sin^2\omega}{\left\{1 + \frac{2b_1}{a_1}\cos\omega + \left(\frac{b_1}{a_1}\right)^2\right\}^{\frac{3}{2}}}$$

$$= \frac{2a_1^2 b_1^2\sin^2\omega}{(a_1^2 + 2a_1b_1\cos\omega + b_1^2)^{\frac{3}{2}}}. \quad (2)$$

Hence if AB, AC (fig. 165) be two tangents to a parabola meeting in A, and CB be bisected in D, since

$$(2AD)^2 = a_1^2 + 2a_1b_1\cos\omega + b_1^2,$$

$$L = \frac{2AB^2\,.\,AC^2\,.\sin^2\omega}{8AD^3} = \frac{(\triangle ABC)^2}{AD^3}. \quad (a)$$

The equation to the curve may be put under the form

$$\left(\sqrt{\frac{c}{f}}x + \sqrt{\frac{a}{f}}y - 1\right)^2 = 4\sqrt{\frac{ac}{f^2}}xy;$$

$$\text{or } \left(\frac{x}{a_1} + \frac{y}{b_1} - 1\right)^2 = \frac{4xy}{a_1 b_1};$$

$$\text{hence } \left(\sqrt{\frac{x}{a_1}} \pm \sqrt{\frac{y}{b_1}}\right) = 1. \quad (\beta)$$

And if $\omega = 90$, the latus rectum $= \dfrac{4a_1^2 b_1^2}{(a_1^2 + b_1^2)^{\frac{3}{2}}}$. $\quad (\gamma)$

20. To find α, β the co-ordinates of the vertex.

$$d'^2 - 4a\phi(\alpha, \beta) = (2bd + 4ae)\alpha; \text{ or } d'^2 = 4bd\alpha;$$

$$e'^2 - 4c\phi(\alpha, \beta) = (2be + 4cd)\beta; \therefore e'^2 = 4be\beta;$$

$$\text{and } -(a + b\cos\omega + c)e' = (b + 2c\cos\omega)d;$$

$$-(a + b\cos\omega + c)d' = (b\cos\omega + 2c)d;$$

$$\therefore \alpha = \frac{d(b\cos\omega + 2c)^2}{4b(a + b\cos\omega + c)^2} = \frac{\frac{d}{a}\left(\frac{2c}{a} + \frac{b}{a}\cos\omega\right)^2}{4\frac{b}{a}\left(1 + \frac{b}{a}\cos\omega + \frac{c}{a}\right)^2},$$

$$\text{or } \alpha = \frac{a_1 b_1^2 (b_1 + a_1\cos\omega)^2}{(a_1^2 + 2a_1 b_1\cos\omega + b_1^2)^2}. \quad (\delta)$$

$$\text{Similarly, } \beta = \frac{b_1 a_1^2 (a_1 + b_1\cos\omega)^2}{(a_1^2 + 2a_1 b_1\cos\omega + b_1^2)^2}. \quad (\epsilon)$$

(ζ) If $\dfrac{a_1}{b_1} = m$, then $\dfrac{\beta}{\alpha} = m\left(\dfrac{m + \cos\omega}{1 + m\cos\omega}\right)^2$; hence if m be constant, $\dfrac{\beta}{\alpha}$ is constant; and if parabolas be drawn touching two given lines AB, AC so that the chord of contact BC is always parallel to a given line, the locus of the vertices of the parabolas is a straight line passing through A.

21. To find α_1, β_1 the co-ordinates of the focus.

$$\alpha_1 = \alpha + \frac{L}{2}\frac{\sin(\omega - \delta)}{\sin\omega} = \alpha + \frac{d\sqrt{ac}\sin^2\omega}{2(a + b\cos\omega + c)^2};$$

$$\text{or } \alpha_1 = \frac{4cd\,(a + b\cos\omega + c)}{4b\,(a + b\cos\omega + c)^2} = \frac{cd}{b\,(a + b\cos\omega + c)}$$

$$= \frac{\dfrac{b_1^2}{a_1^2}\,.\,2b_1}{\dfrac{2b_1}{a_1}\left\{1 + \dfrac{2b_1}{a_1}\cos\omega + \dfrac{b_1^2}{a_1^2}\right\}};$$

$$\text{hence } \alpha_1 = \frac{b_1^2\,a_1}{a_1^2 + 2a_1b_1\cos\omega + b_1^2}. \;\ldots\ldots\ldots\ldots \;(1)$$

$$\text{Similarly, } \beta_1 = \frac{a_1^2\,b_1}{a_1^2 + 2a_1b_1\cos\omega + b_1^2}, \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{and } \frac{\beta_1}{\alpha_1} = \frac{a_1}{b_1};\; \text{ or if } \frac{a_1}{b_1} = m;\; \frac{\beta_1}{\alpha_1} = m;\; \ldots\ldots\ldots \;(3)$$

hence if a series of parabolas be drawn to touch two given straight lines so that the chord of contact may be always parallel to a given line, the locus of the foci is a straight line passing through A.

22. If S (fig. 165) be the focus, and $AS = \rho_1$, bisect BC in D,

$$\text{then } \rho_1^2 = \alpha_1^2 + 2\alpha_1\beta_1\cos\omega + \beta_1^2$$

$$= (b_1^2 + 2a_1b_1\cos\omega + a_1^2)\frac{(a_1b_1)^2}{(a_1^2 + 2a_1b_1\cos\omega + b_1^2)^2},$$

$$\text{or } \rho_1^2 = \frac{a_1^2b_1^2}{a_1^2 + 2a_1b_1\cos\omega + b_1^2} = a_1\alpha_1 = \beta_1 b_1;\; \ldots\ldots \;(4)$$

$$\text{hence } 4AS^2 = \frac{(AB\,.\,AC)^2}{AD^2} \text{ or } 2AS\,.\,AD = AB\,.\,AC;\; \ldots\;(5)$$

and if SE, SF be drawn parallel to AC, AB respectively; $AS^2 = AE\,.\,AB = AF\,.\,AC$ and a circle may be described round the points B, E, F, C. .... (6)

23. When a parabola slides between two straight lines inclined at a given angle ω, to find the locus of the focus.

$$L = \frac{2a_1^2b_1^2\sin^2\omega}{(a_1^2 + 2a_1b_1\cos\omega + b_1^2)^{\frac{3}{2}}} = \frac{2\rho_1^3\sin^2\omega}{a_1b_1} = \frac{2\alpha_1\beta_1\sin^2\omega}{\rho_1};$$

$$\therefore\ \frac{\rho_1^2}{\alpha_1^2\beta^2} = \frac{4\sin^4\omega}{L^2}; \quad \therefore\ \frac{1}{\alpha_1^2} + \frac{2\cos\omega}{\alpha_1\beta_1} + \frac{1}{\beta_1^2} = \frac{4\sin^4\omega}{L^2}$$

is the equation required.

24. When a parabola slides between two straight lines inclined at a given angle ω, to find the locus of the vertex.

Let x, y be the co-ordinates of the vertex in any position;

$$\text{then } x = \frac{a_1 b_1^2 (b_1 + a_1\cos\omega)^2}{r_1^4}; \quad y = \frac{b_1 a_1^2 (a_1 + b_1\cos\omega)^2}{r_1^4};$$

$$L = \frac{2a_1^2 b_1^2 \sin^2\omega}{r_1^3};$$

where $r_1^2 = a_1^2 + 2a_1 b_1\cos\omega + b_1^2$; and if $\dfrac{b_1}{a_1} = m$, we have

$$\frac{x}{L} = \frac{(m+\cos\omega)^2}{2\sin^2\omega\,(1+2m\cos\omega+m^2)^{\frac{1}{2}}};$$

$$\frac{y}{L} = \frac{(1+m\cos\omega)^2}{2\sin^2\omega\, m\sqrt{1+2m\cos\omega+m^2}};$$

$$\text{or } \frac{2x}{a} = \frac{(m+\cos\omega)^2}{\sqrt{1+2m\cos\omega+m^2}};$$

$$\frac{2y}{a} = \frac{\left(\frac{1}{m}+\cos\omega\right)^2}{\sqrt{\left(\frac{1}{m}+\cos\omega\right)^2+\sin^2\omega}};$$

$$\text{where } a = \frac{L}{\sin^2\omega};$$

$$\text{hence } m + \cos\omega = \sqrt{2\left(\frac{x^2}{a^2} + \frac{x}{a}\sqrt{\frac{x^2}{a^2}+\sin^2\omega}\right)};$$

$$\frac{1}{m} + \cos\omega = \sqrt{2\left(\frac{y^2}{a^2} + \frac{y}{a}\sqrt{\frac{y^2}{a^2}+\sin^2\omega}\right)};$$

$$\therefore \left\{ \sqrt{\frac{2x}{a}\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} + \sin^2\omega}\right)} - \cos\omega \right\}$$

$$\times \left\{ \sqrt{\frac{2y}{a}\left(\frac{y}{a} + \sqrt{\frac{y^2}{a^2} + \sin^2\omega}\right)} - \cos\omega \right\} = 1.$$

25. To find the equation to a conic section passing through five given points A, B, C, D, E.

Produce BA, CD (fig. 166) to meet in O; and let OAB, ODC be the co-ordinate axes; $OA = a$, $OD = b$, $OB = a'$, $OC = b'$; then the equation to the conic section is

$$\lambda xy + \left(\frac{x}{a} + \frac{y}{b} - 1\right)\left(\frac{x}{a'} + \frac{y}{b'} - 1\right) = 0;$$

and in order that this may pass through the point E whose co-ordinates are a'', b'', we have

$$\lambda a'' b'' + \left(\frac{a''}{a} + \frac{b''}{b} - 1\right)\left(\frac{a''}{a'} + \frac{b''}{b'} - 1\right) = 0,$$

or $\left(\frac{a''}{a}+\frac{b''}{b}-1\right)\left(\frac{a''}{a'}+\frac{b''}{b'}-1\right)xy - a''b''\left(\frac{x}{a}+\frac{y}{b}-1\right)\left(\frac{x}{a'}+\frac{y}{b'}-1\right) = 0,$

is the equation required.

Cor. Hence may be found the equation to the parabola which passes through four fixed points.

The general equation to the conic section is

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right)\left(\frac{x}{a'} + \frac{y}{b'} - 1\right) + \lambda xy = 0;$$

or $Ay^2 + Bxy + Cx^2 + Dy + Ex + 1 = 0$; and if λ be assumed so that $B^2 = 4AC$, we have

$$\frac{x^2}{aa'} \pm \frac{2xy}{\sqrt{aa'bb'}} + \frac{y^2}{bb'} - \left(\frac{1}{a} + \frac{1}{a'}\right)x - \left(\frac{1}{b} + \frac{1}{b'}\right)y + 1 = 0,$$

which is the equation required.

Also the equation to a line through the origin parallel to the axis of the parabola is $y + \frac{2C}{B}x = 0$, (Art. 11, Equation 6).

Therefore the line $y = \pm\sqrt{\frac{bb'}{aa'}}\,x$ is the equation to a line through the origin parallel to the diameters of the two parabolas.

26. To find the locus of the centres of all the curves of the second order which pass through four given points.

The equation to the conic section is

$$\lambda xy + \left(\frac{x}{a} + \frac{y}{b} - 1\right)\left(\frac{x}{a'} + \frac{y}{b'} - 1\right) = 0;$$

and if h, k be the co-ordinates of the centre, the equation from the centre becomes

$$\lambda(x'+h)(y'+k) + \left\{\frac{x'}{a} + \frac{y'}{b} + \left(\frac{h}{a} + \frac{k}{b} - 1\right)\right\}\left\{\frac{x'}{a'} + \frac{y'}{b'} + \left(\frac{h}{a'} + \frac{k}{b'} - 1\right)\right\} = 0;$$

$$\text{or } \lambda h + \frac{1}{b}\left(\frac{h}{a'} + \frac{k}{b'} - 1\right) + \frac{1}{b'}\left(\frac{h}{a} + \frac{k}{b} - 1\right) = 0;$$

$$\lambda k + \frac{1}{a}\left(\frac{h}{a'} + \frac{k}{b'} - 1\right) + \frac{1}{a'}\left(\frac{h}{a} + \frac{k}{b} - 1\right) = 0;$$

$$\therefore \left(\frac{k}{b} - \frac{h}{a}\right)\left(\frac{h}{a'} + \frac{k}{b'} - 1\right) + \left(\frac{k}{b'} - \frac{h}{a'}\right)\left(\frac{h}{a} + \frac{k}{b} - 1\right) = 0;$$

$$\text{or } \frac{2k^2}{bb'} - \frac{2h^2}{aa'} - k\left(\frac{1}{b} + \frac{1}{b'}\right) + h\left(\frac{1}{a} + \frac{1}{a'}\right) = 0; \quad (a)$$

which is the equation to a conic section, the co-ordinates of whose centre are $\frac{a+a'}{4}$, $\frac{b+b'}{4}$.

If $\frac{b}{a} = \frac{b'}{a'}$, the equation becomes a straight line

$$\frac{1}{b}\left(\frac{h}{a'} + \frac{k}{b'} - 1\right) + \frac{1}{b'}\left(\frac{h}{a} + \frac{k}{b} - 1\right) = 0; \text{ or}$$

$$\frac{2h}{ba'} + \frac{2k}{bb'} - \frac{b+b'}{bb'} = 0; \qquad \therefore \frac{2h}{a+a'} + \frac{2k}{b+b'} = 1. \quad (\beta)$$

27. If $u = 0$, $v = 0$, $\omega = 0$ be the equations to three straight lines, then $\sqrt{u} + \sqrt{\alpha v} + \sqrt{\beta\omega} = 0$ will be the equation to a conic section which touches the three straight lines, where α and β are arbitrary constants.

Since $\sqrt{u} + \sqrt{\alpha v} + \sqrt{\beta\omega} = 0$,

$$u = \alpha v + \beta\omega - 2\sqrt{\alpha\beta}\sqrt{v\omega};$$

$$\therefore (u - \alpha v - \beta\omega)^2 = 4\alpha\beta v\omega, \text{ or}$$

$$u^2 + \alpha^2 v^2 + \beta^2\omega^2 - 2\alpha uv - 2\beta u\omega - 2\alpha\beta v\omega = 0;$$

and when $u = 0$, $\alpha^2v^2 + \beta^2\omega^2 - 2\alpha\beta v\omega = 0$; or $\alpha v - \beta\omega = 0$; therefore the straight line $u = 0$ will meet the conic section in the straight line whose equation is $\alpha v - \beta\omega = 0$; or in one point only; hence $u = 0$ touches the conic section.

Similarly the straight lines $v = 0$, $\omega = 0$ will touch the conic section.

28. If $u = 0$, $v = 0$, $\omega = 0$ be the equations to the three lines BC, AC, AB (fig. 196), which touch a conic section in the points a, b, c respectively; since $\alpha v - \beta\omega = 0$ is the equation to a straight line passing through A the intersection of $v = 0$, $\omega = 0$, and also through the point of contact with $u = 0$, it is the equation to Aa.

Similarly the equations to Bb, Cc are $u - \beta\omega = 0$, $u - \alpha v = 0$; and at the point of intersection of Bb, Cc, $(u - \beta\omega) - (u - \alpha v) = 0$, or $\alpha v - \beta\omega = 0$, which is a point in Aa; therefore Aa, Bb, Cc meet in the same point.

29. To find the condition that the straight line whose equation is $u + Av + B\omega = 0$, may touch the conic section whose equation is $\sqrt{u} + \sqrt{\alpha v} + \sqrt{\beta\omega} = 0$.

We have $u = \alpha v + \beta\omega + 2\sqrt{\alpha\beta}\sqrt{v\omega}$; hence at the points of intersection with the line $u + Av + B\omega = 0$,

$$(A + \alpha)v + (B + \beta)\omega + 2\sqrt{\alpha\beta}\sqrt{v\omega} = 0; \qquad (2)$$

this will in general, when combined with $u + Av + B\omega = 0$, give two values of x, and intersect the curve in two points; but when equation (2) is a complete square,

$$\sqrt{(A + \alpha)v} + \sqrt{(B + \beta)\omega} = 0, \text{ or } (A + \alpha)v = (B + \beta)\omega;$$

therefore x is determined by a simple equation and has only one value; in which case equation (1) becomes that of a tangent. Hence $(A + \alpha)(B + \beta) = \alpha\beta$; or $\dfrac{\alpha}{A} + \dfrac{\beta}{B} + 1 = 0$, which is the condition required*.

30. To find the condition that $\dfrac{x}{\alpha} + \dfrac{y}{\beta} - 1 = 0$ may be the equation to a tangent to the curve

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

referred to two tangents as axes.

$$a\beta^2\left(\frac{y}{\beta}\right)^2 + b\alpha\beta\left(\frac{x}{\alpha}\frac{y}{\beta}\right) + c\alpha^2\left(\frac{x}{\alpha}\right)^2 + d\beta\left(\frac{y}{\beta}\right) + e\alpha\left(\frac{x}{\alpha}\right) + f = 0;$$

$$\therefore\ a\beta^2\left(1-\frac{x}{\alpha}\right)^2 + b\alpha\beta\left(\frac{x}{\alpha}\right)\left(1-\frac{x}{\alpha}\right) + c\alpha^2\left(\frac{x}{\alpha}\right)^2 + d\beta\left(1-\frac{x}{\alpha}\right) + e\alpha\frac{x}{\alpha} + f = 0;$$

$$\text{or } (a\beta^2 - b\alpha\beta + c\alpha^2)\left(\frac{x}{\alpha}\right)^2 - (2a\beta^2 - b\alpha\beta + d\beta - e\alpha)\frac{x}{\alpha} + a\beta^2 + d\beta + f = 0; \quad (1)$$

and in order that $\dfrac{x}{\alpha} + \dfrac{y}{\beta} - 1 = 0$ may be a tangent, $\dfrac{x}{\alpha}$ must only have one value; and equation (1) must be a square;

* This condition is given by Mr Hearn in his "Researches on Curves of the second order."

$\therefore\ (2a\beta^2 - ba\beta + d\beta - e\alpha)^2 = 4(a\beta^2 - ba\beta + c\alpha^2)(a\beta^2 + d\beta + f)$;

or $(b^2 - 4ac)\alpha^2\beta^2 - 2(2ae - bd)\alpha\beta^2 - 2(2cd - be)\alpha^2\beta$

$$+ 2(2bf - de)\alpha\beta + (d^2 - 4af)\beta^2 + (e^2 - 4cf)\alpha^2 = 0;$$

and when the axes of x and y are tangents $d^2 - 4af = 0$, $e^2 - 4cf = 0$;

$$\therefore\ \alpha\beta - 2h\beta - 2k\alpha + 2g = 0, \quad \ldots\ldots\ldots\ldots \quad (2)$$

which is the condition required.

When $b^2 - 4ac = 0$, we have $H\beta + K\alpha - G = 0$. ... (3)

But $H\beta + K\alpha - G = 0$ is the equation to the chord of contact BC (Art. 61); hence if from any point D in the chord of contact of a parabola, (fig. 167) DE, DF be drawn parallel to the tangents to meet them in E, F; EF will be a tangent to the parabola.

31. If a conic section touches four given straight lines, viz. the axes of x, y, and the two straight lines $\frac{x}{\alpha} + \frac{y}{\beta} - 1 = 0$, $\frac{x}{\alpha'} + \frac{y}{\beta'} - 1 = 0$; to find the locus of the centre.

From equation (2) we have

$$\alpha\beta - 2h\beta - 2k\alpha + 2g = 0;$$

$$\alpha'\beta' - 2h\beta' - 2k\alpha' + 2g = 0;$$

$$\therefore\ (\alpha'\beta' - \alpha\beta) - 2(\beta' - \beta)h - 2(\alpha' - \alpha)k = 0; \qquad (4)$$

which gives a relation between h and k, and is evidently that of a straight line passing through the points $\frac{\alpha}{2}, \frac{\beta'}{2}$; $\frac{\alpha'}{2}, \frac{\beta}{2}$; or the straight line joining the middle points of the diagonals of the quadrilateral figure formed by the intersection of the four straight lines.

(a) Hence if a conic section touches five lines α, β, γ, δ, ϵ, let any four of them, as α, β, γ, δ form a quadrilateral figure q_1; the centre will be in the line δ_1 joining the middle points of the diagonals of q_1; similarly, let ϵ and any three of the others, as α, β, γ form a quadrilateral figure q_2; the centre will be in

the line δ_2 joining the middle points of the diagonals of q_2; therefore the intersection of δ_1, δ_2 will be the centre of the conic section.

32. From Art. 11. If x, y be the co-ordinates of the focus of a parabola,

$$Kx + Hy - G = -2Ky\cos\omega\,; \quad \text{and } Hx = Ky\,;$$

$$\text{and } K\alpha + H\beta - G = 0, \quad \text{(Equation 3, Art. 30)};$$

$$\therefore\ K(x-\alpha) + H(y-\beta) = -2Ky\cos\omega,$$

$$\text{or } \frac{x}{y}(x-\alpha) + y - \beta = -2x\cos\omega\,;$$

$$\therefore\ x^2 + 2\cos\omega\, xy + y^2 - \alpha x - \beta y = 0, \quad \ldots\ldots \quad (5)$$

which is the equation to a circle. Hence the locus of the foci of all parabolas which touch three given straight lines is a circle passing through their points of intersection.

33. (α) From equation (2) it appears that if any tangent EF (fig. 167) be drawn between two given tangents AB, AC to a curve of the second order which has a centre, and the parallelogram $AEDF$ be completed, the locus of D will be a hyperbola whose asymptotes are parallel to AB, AC.

From equation (3) it appears that if the curve be a parabola, the locus of D will be a straight line, viz. the line joining the points of contact of the two tangents AB, AC.

(β) Since $\frac{\alpha}{2}$, $\frac{\beta}{2}$ are the co-ordinates of Q the middle point of EF, when the curve has a centre, the locus of Q is a hyperbola; but when the curve is a parabola, the locus of Q is a straight line.

34. To find the equation to a conic section which shall touch five given straight lines.

Let two of the straight lines be taken for the co-ordinate axes, and let

$$\frac{x}{a} + \frac{y}{b} - 1 = 0, \quad \frac{x}{a'} + \frac{y}{b'} - 1 = 0, \quad \frac{x}{a''} + \frac{y}{b''} - 1 = 0,$$

or $u = 0$, $u' = 0$, $u'' = 0$ be the equations to the three remaining straight lines; then the equation to the conic section is $\sqrt{u} + \sqrt{\alpha u'} + \sqrt{\beta u''} = 0$; where α and β are constants to be determined.

$$\text{let } \frac{1}{a} + \frac{A}{a'} + \frac{B}{a''} = \lambda,\ \frac{1}{b} + \frac{A}{b'} + \frac{B}{b''} = 0,\ 1 + A + B = 0;$$

$$\therefore\ u + Au' + Bu'' = \lambda x, \text{ and } A = \frac{\frac{1}{b} - \frac{1}{b''}}{\frac{1}{b''} - \frac{1}{b'}};\ B = \frac{\frac{1}{b'} - \frac{1}{b}}{\frac{1}{b''} - \frac{1}{b'}};$$

$$\text{Similarly, if } A' = \frac{\frac{1}{a} - \frac{1}{a''}}{\frac{1}{a''} - \frac{1}{a'}},\ B' = \frac{\frac{1}{a'} - \frac{1}{a}}{\frac{1}{a''} - \frac{1}{a'}},$$

$$u + A'u' + B'u'' = \lambda' y;$$

and when the axes of x and y are tangents, (Art. 29)

$$\frac{\alpha}{A'} + \frac{\beta}{B'} + 1 = 0,\ \frac{\alpha}{A} + \frac{\beta}{B} + 1 = 0,$$

$$\text{or } \frac{\alpha}{\left(\frac{1}{a} - \frac{1}{a''}\right)} + \frac{\beta}{\frac{1}{a'} - \frac{1}{a}} + \frac{1}{\left(\frac{1}{a''} - \frac{1}{a'}\right)} = 0,$$

$$\frac{\alpha}{\left(\frac{1}{b} - \frac{1}{b''}\right)} + \frac{\beta}{\left(\frac{1}{b'} - \frac{1}{b}\right)} + \frac{1}{\left(\frac{1}{b''} - \frac{1}{b'}\right)} = 0,$$

$$\text{and if } \alpha = \alpha' \left(\frac{1}{a} - \frac{1}{a''}\right)\left(\frac{1}{b} - \frac{1}{b''}\right),$$

$$\beta = \beta' \left(\frac{1}{a'} - \frac{1}{a}\right)\left(\frac{1}{b'} - \frac{1}{b}\right),\quad \left(\frac{1}{a''} - \frac{1}{a'}\right)\left(\frac{1}{b''} - \frac{1}{b'}\right) = \frac{1}{P};$$

$$\therefore\ \alpha' \left(\frac{1}{b} - \frac{1}{b''}\right) + \beta' \left(\frac{1}{b'} - \frac{1}{b}\right) + P \left(\frac{1}{b''} - \frac{1}{b'}\right) = 0,$$

$$\alpha' \left(\frac{1}{a} - \frac{1}{a''}\right) + \beta' \left(\frac{1}{a'} - \frac{1}{a}\right) + P \left(\frac{1}{a''} - \frac{1}{a'}\right) = 0;$$

$$\text{or } (\alpha' - P)\left(\frac{1}{b} - \frac{1}{b''}\right) + (\beta' - P)\left(\frac{1}{b'} - \frac{1}{b}\right) = 0,$$

$$(a' - P)\left(\frac{1}{a} - \frac{1}{a''}\right) + (\beta' - P)\left(\frac{1}{a'} - \frac{1}{a}\right) = 0;$$

$$\therefore\ a' - P = 0,\ \beta' - P = 0,\ \text{or } a' = P,\ \beta' = P,$$

$$\therefore\ a = \frac{\left(\frac{1}{a} - \frac{1}{a''}\right)\left(\frac{1}{b} - \frac{1}{b''}\right)}{\left(\frac{1}{a'} - \frac{1}{a''}\right)\left(\frac{1}{b'} - \frac{1}{b''}\right)},\quad \beta = \frac{\left(\frac{1}{b} - \frac{1}{b'}\right)\left(\frac{1}{a} - \frac{1}{a'}\right)}{\left(\frac{1}{a''} - \frac{1}{a'}\right)\left(\frac{1}{b''} - \frac{1}{b'}\right)};$$

$$\text{hence } \sqrt{\left(\frac{1}{a'} - \frac{1}{a''}\right)\left(\frac{1}{b'} - \frac{1}{b''}\right)\left(\frac{x}{a} + \frac{y}{b} - 1\right)}$$

$$+ \sqrt{\left(\frac{1}{a} - \frac{1}{a''}\right)\left(\frac{1}{b} - \frac{1}{b''}\right)\left(\frac{x}{a'} + \frac{y}{b'} - 1\right)}$$

$$+ \sqrt{\left(\frac{1}{a'} - \frac{1}{a}\right)\left(\frac{1}{b'} - \frac{1}{b}\right)\left(\frac{x}{a''} + \frac{y}{b''} - 1\right)} = 0,$$

is the equation required.

35. Find the equation to the conic section which passes through a given point, and touches four given straight lines.

Let two of the given straight lines be taken for the co-ordinate axes; and let $\frac{x}{a} + \frac{y}{b} - 1 = 0$, $\frac{x}{a'} + \frac{y}{b'} - 1 = 0$, be the equations to the two remaining straight lines; and a'', b'' the co-ordinates of the given point; then the equation to the conic section is

$$\sqrt{\frac{x}{a} + \frac{y}{b} - 1} + \sqrt{\alpha x} + \sqrt{\beta y} = 0, \dots\dots\dots\dots \quad (1)$$

$$\text{and } \frac{x}{a'} + \frac{y}{b'} - 1 = \left(\frac{x}{a} + \frac{y}{b} - 1\right) + Ax + By;$$

$$\therefore\ \frac{1}{a'} = \frac{1}{a} + A,\quad \frac{1}{b'} = \frac{1}{b} + B,$$

$$\text{or } A = \left(\frac{1}{a'} - \frac{1}{a}\right),\quad B = \frac{1}{b'} - \frac{1}{b};$$

$$\text{hence } \frac{\alpha}{A} + \frac{\beta}{B} + 1 = 0;\ \text{(Art. 29)}.$$

$$\text{and } \sqrt{\alpha a''} + \sqrt{\beta b''} + \sqrt{\frac{a''}{a} + \frac{b''}{b} - 1} = 0;$$

from which equations α and β are determined, and these values when substituted in equation (1) will give the equation required.

There will in general be two conic sections corresponding to the two values of α and β; but when $\frac{a''}{a} + \frac{b''}{b} - 1 = 0$, or the point a'', b'' is in one of the given straight lines, α, β have only one value respectively.

36. Find the equation to the conic section which passes through two points, and touches three given straight lines.

$$\text{Let } \sqrt{\frac{x}{a} + \frac{y}{b} - 1} + \sqrt{\alpha x} + \sqrt{\beta y} = 0, \quad (1)$$

be the equation to the conic section; and a', b'; a'', b'' the co-ordinates of the given points; then

$$\sqrt{\frac{a'}{a} + \frac{b'}{b} - 1} + \sqrt{\alpha a'} + \sqrt{\beta b'} = 0;$$

$$\sqrt{\frac{a''}{a} + \frac{b''}{b} - 1} + \sqrt{\alpha a''} + \sqrt{\beta b''} = 0;$$

from which α, β may be determined; and equation (1) is the equation required.

36.* Find the equation to a conic section which passes through three points and touches two given straight lines.

Let the two given straight lines be taken for the co-ordinate axes, then if $\frac{x}{\alpha} + \frac{y}{\beta} - 1 = 0$, be the equation to the chord of contact of the two tangents, the equation to the conic section is $\left(\frac{a}{x} + \frac{y}{\beta} - 1\right)^2 + \lambda xy = 0$, where α, β, λ are to be determined. If a, b; a', b'; a'', b''; be the given points,

$$\left(\frac{a}{\alpha}+\frac{b}{\beta}-1\right)^2+\lambda ab=0, \quad \left(\frac{a'}{\alpha}+\frac{b'}{\beta}-1\right)^2+\lambda a'b'=0\,;$$

$$\text{hence } \frac{a}{\alpha}+\frac{b}{\beta}-1=\sqrt{\frac{ab}{a'b'}}\left(\frac{a'}{\alpha}+\frac{b'}{\beta}-1\right),$$

$$\text{or } \frac{\sqrt{aa'}}{\alpha}-\frac{\sqrt{bb'}}{\beta}=\frac{\sqrt{a'b'}-\sqrt{ab}}{\sqrt{ab'}-\sqrt{a'b}}\,. \quad \ldots\ldots\ldots\ldots(1)$$

$$\text{Similarly, } \frac{\sqrt{aa''}}{\alpha}-\frac{\sqrt{bb''}}{\beta}=\frac{\sqrt{a''b''}-\sqrt{ab}}{\sqrt{ab''}-\sqrt{a''b}}\,; \quad \ldots\ldots\ldots\ (2)$$

from which equations $\frac{1}{\alpha}$, and $\frac{1}{\beta}$ may be determined, and the equation to the conic section is

$$ab\left(\frac{x}{\alpha}+\frac{y}{\beta}-1\right)^2-\left(\frac{a}{\alpha}+\frac{b}{\beta}-1\right)^2 xy=0.$$

From equation (1) it appears that when a conic section passes through two given points, and touches two given lines AB, AC; then if BC be a chord of contact, and the parallelogram $ABPC$, (fig. 167) be completed, the locus of P is a hyperbola; unless $ab=a'b'$; in which case $\beta=\sqrt{\frac{bb'}{aa'}}\,\alpha$ and the locus is a straight line passing through the origin.

The locus of the middle point of the chord of contact BC is a hyperbola.

37. Find the equation to a conic section which shall pass through four given points, and touch a given straight line.

Let the given straight line be taken for the axis of x; and let the equations to the four lines joining the given points be

$$\frac{x}{a}+\frac{y}{b}-1=u=0\,; \quad \frac{x}{a'}+\frac{y}{b'}-1=u'=0, \text{ \&c.};$$

then the equation to the conic section is

$$uu''+\lambda u'u'''=0=Ay^2+Bxy+Cx^2+Dy+Ex+F\,;$$

and since the axis of x is a tangent, $E^2=4CF$,

but $C=\frac{1}{aa''}+\frac{\lambda}{a'a'''}$; $-E=\left(\frac{1}{a}+\frac{1}{a''}\right)+\lambda\left(\frac{1}{a'}+\frac{1}{a'''}\right)$, $F=1+\lambda$;

$$\therefore \left\{\left(\frac{1}{a}+\frac{1}{a''}\right)+\lambda\left(\frac{1}{a'}+\frac{1}{a'''}\right)\right\}^2 = 4\left\{\frac{1}{aa''}+\frac{\lambda}{a'a'''}\right\}\{1+\lambda\};$$

$$\therefore \left(\frac{1}{a'}-\frac{1}{a'''}\right)^2\lambda^2+2\left\{\left(\frac{1}{a}+\frac{1}{a''}\right)\left(\frac{1}{a'}+\frac{1}{a'''}\right)-2\left(\frac{1}{a'a'''}+\frac{1}{aa''}\right)\right\}\lambda+\left(\frac{1}{a}-\frac{1}{a''}\right)^2=0;$$

$$\therefore \left(\frac{1}{a'}-\frac{1}{a'''}\right)^2\lambda^2+2\left(\frac{1}{aa'}+\frac{1}{aa'''}+\frac{1}{a'a''}+\frac{1}{a''a'''}-\frac{2}{aa''}-\frac{2}{a'a'''}\right)\lambda+\left(\frac{1}{a}-\frac{1}{a''}\right)^2=0;$$

$$\therefore \left(\frac{1}{a'}-\frac{1}{a'''}\right)^2\lambda^2+2\left\{\left(\frac{1}{a'}-\frac{1}{a'''}\right)\left(\frac{1}{a}-\frac{1}{a''}\right)+2\left(\frac{1}{a'}-\frac{1}{a}\right)\left(\frac{1}{a''}-\frac{1}{a'''}\right)\right\}\lambda+\left(\frac{1}{a}-\frac{1}{a''}\right)^2=0;$$

$$\text{or } \left(\frac{1}{a'}-\frac{1}{a'''}\right)^2\lambda+\left(\frac{1}{a'}-\frac{1}{a'''}\right)\left(\frac{1}{a}-\frac{1}{a''}\right)+2\left(\frac{1}{a'}-\frac{1}{a}\right)\left(\frac{1}{a''}-\frac{1}{a'''}\right)$$

$$= \pm 2\sqrt{\left(\frac{1}{a'}-\frac{1}{a}\right)\left(\frac{1}{a''}-\frac{1}{a'''}\right)\left(\frac{1}{a'}-\frac{1}{a''}\right)\left(\frac{1}{a}-\frac{1}{a''}\right)}$$

which determines two values of λ.

Since $\lambda\lambda' = \dfrac{\left(\frac{1}{a}-\frac{1}{a''}\right)^2}{\left(\frac{1}{a'}-\frac{1}{a'''}\right)^2}$; there are evidently two curves whose equations are $uu''+\lambda u'u'''=0$;

$$\text{and } uu''+\frac{\left(\frac{1}{a}-\frac{1}{a''}\right)^2}{\left(\frac{1}{a'}-\frac{1}{a'''}\right)^2\lambda}\,.\,u'u'''=0.$$

37. (*a*) To find the points of contact, when a conic section passes through four given points, and touches a given line.

Let the four lines joining the given points meet the tangent in P_1, P_2, P_3, P_4 respectively (fig. 188); and let A be the required point of contact; then (Art. 54.) if A be supposed for convenience to the left of P_1,

$$\frac{1}{AP_1} + \frac{1}{AP_3} = \frac{1}{AP_2} + \frac{1}{AP_4};$$

$$\text{or } \frac{1}{AP_1} - \frac{1}{AP_2} = \frac{1}{AP_4} - \frac{1}{AP_3};$$

$$\text{hence } \frac{P_1P_2}{AP_1 \, . \, AP_2} = \frac{P_3P_4}{AP_3 \, . \, AP_4},$$

or if $AP_1 = x$, $P_1P_2 = a_2$, $P_1P_3 = a_3$, $P_1P_4 = a_4$;

$$\frac{a_2}{x(x + a_2)} = \frac{a_3 - a_4}{(x + a_3)(x + a_4)};$$

$$\therefore \; x = \frac{-a_2a_4}{a_2 + a_4 - a_3} \pm \frac{\sqrt{a_2a_4(a_2 - a_3)(a_4 - a_3)}}{(a_2 + a_4 - a_3)},$$

from which the two positions of A may easily be determined by a geometrical construction; and the problem is reduced to that of the determination of two conic sections each of which passes through five given points.

If x be negative, A will be to the right of P_1.

37. (β) Having given two tangents to a parabola and the direction of the axis, find the focus and vertex.

Let PT, QT (fig. 210) be the two tangents, PM the direction of the axis; draw QN parallel to PM, and make $\angle SPT = \angle TPM$, and $\angle TQS = \angle TQN$; then PS, QS will intersect in S the focus of the parabola.

Draw SY, SZ perpendicular to TP, TQ respectively; then Y and T are points in the tangent at the vertex; join YZ and draw SV perpendicular to YZ; V will be the vertex required.

37. (γ) To describe the two parabolas which pass through four given points.

Let A, B, C, D (fig. 211) be the four given points, produce AD, BC to meet in G', and BA, CD to meet in F; join BD, AC; then a pair of tangents drawn from A and D intersect in FE; (Art. 62); and Art. 25. Cor. 1. $y = \pm \sqrt{\dfrac{bb'}{aa'}}\, x$

is the equation to the two lines parallel to the diameters of the two parabolas; hence if $FK = FK'$ be a mean proportional between FA and FB, and FL a mean proportional between FD and FC, the axes of the two parabolas will be parallel to KL, and $K'L$; bisect AD in G; draw GH parallel to $K'L$ meeting EF in H; then HD, HA are tangents, and if GH be bisected in P, P will be a point in the parabola; and since the tangents HD, HA at the points D, A, and $K'L$ the direction of the axis of the parabola is known, the focus and vertex may be determined 37 (β).

Similarly the parabola may be described whose axis is parallel to LK.

37. (δ) To describe a parabola touching four given straight lines.

Let AB, BC, CD, DA (fig. 212) be the four given straight lines; produce BA, CD to meet in E; and AD, BC to meet in F; about EAD, FCD describe two circles; these will intersect each other in S the focus of the parabola (Art. 32); from S draw two perpendiculars SY, SZ on EA, ED respectively; then YZ will be a tangent at the vertex; and if SV be drawn perpendicular to YZ, V will be the vertex required.

38. Let $y + m_1 x + c_1 = 0$, $y + m_2 x + c_2 = 0$,

$$y + m_3 x + c_3 = 0, \quad y + m_4 x + c_4 = 0$$

be the equations to four sides of a quadrilateral figure inscribed in a conic section whose equation is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

then the equation to the conic section may be put under the form

$$(y + m_1 x + c_1)(y + m_3 x + c_3) + \lambda (y + m_2 x + c_2)(y + m_4 x + c_4) = 0; \text{ (A)}$$

$$\text{or } (1 + \lambda) y^2 + \{(m_1 + m_3) + \lambda (m_2 + m_4)\} xy$$
$$+ (m_1 m_3 + \lambda m_2 m_4) x^2 + \{c_1 + c_3 + \lambda (c_2 + c_4)\} y$$
$$+ \{m_1 c_3 + m_3 c_1 + \lambda (m_2 c_4 + m_4 c_2)\} x + c_1 c_3 + \lambda c_2 c_4 = 0;$$

and in order that this may coincide with the given equation to the curve, we must have the five following relations:

$$a\left\{(m_1 + m_3) + \lambda(m_2 + m_4)\right\} = b(1 + \lambda); \tag{1}$$

$$a(m_1 m_3 + \lambda m_2 m_4) = c(1 + \lambda); \tag{2}$$

$$a\left\{c_1 + c_3 + \lambda(c_2 + c_4)\right\} = d(1 + \lambda); \tag{3}$$

$$a\left\{c_1 m_3 + c_3 m_1 + \lambda(c_2 m_4 + c_4 m_2)\right\} = e(1 + \lambda); \tag{4}$$

$$a(c_1 c_3 + \lambda c_2 c_4) = f(1 + \lambda). \tag{5}$$

Equation (A) equally represents the equation to the conic section when

$$y + m_2 x + c_2 = 0, \quad y + m_4 x + c_4 = 0,$$

are the equations to the diagonals of the quadrilateral.

39. If any three of the quantities m_1, m_2, m_3, m_4 be given, the fourth may be determined by equations (1), (2); hence if three sides of a quadrilateral figure inscribed in a conic section be parallel to three fixed lines, the fourth side will also be parallel to a fixed line.

40. In a conic section inscribe a quadrilateral figure, such that one of its sides may pass through a fixed point, and its three remaining sides be parallel to three fixed lines.

Let P (fig. 168) be the fixed point; draw any line AB parallel to the first side, and BC, CD respectively parallel to the second and third sides; then the remaining side DA of the quadrilateral figure will always be parallel to a fixed line. Hence, through P draw $PD'A'$ parallel to DA, and $A'B'$, $B'C'$ parallel to AB, BC; the remaining $C'D'$ will be parallel to CD, and $A'B'C'D'$ will be the figure required.

41. In a conic section inscribe a triangle similar and similarly situated to a given triangle.

Describe as before a quadrilateral figure whose sides AB, BC, CD (fig. 169) shall be parallel to the three sides of the triangle; join DA; then DA is parallel to a fixed line: draw

any other line $A'D'$ parallel to AD, and bisect AD, $A'D'$ by the straight line aa'; draw ab, bc parallel to AB, BC; then ac will be parallel to CD; for abc may be considered as a quadrilateral figure whose evanescent side at a coincides with the tangent at a; and since the tangent at a, ab, bc are parallel to DA, AB, BC, the fourth side ac will be parallel to CD.

42. If $ABCDEF$ (fig. 170) be a hexagon inscribed in a conic section, draw the diagonal AD, and let DE, EF be parallel to AB, BC respectively; then $ABCD$, $DEFA$ are two quadrilateral figures, having three of their sides DA, AB, BC, and AD, DE, EF parallel to the same three lines respectively; hence the remaining sides AF, CD will be parallel to a fixed line; and therefore to one another.

43. If three of the quantities c_1, c_2, c_3, c_4 be given, the fourth may be determined by equations (3), (5); hence if three sides of a quadrilateral figure inscribed in a conic section pass through three fixed points in the axis of y, the fourth side will pass through a fixed point in the axis of y; and the axis of y may be taken in any direction; hence if three sides of a quadrilateral inscribed in a conic section pass through three fixed points in a straight line, the fourth side will pass through a fixed point in the same straight line.

44. If $m_1 + m_3 = \frac{b}{a}$; equation (1) gives $m_2 + m_4 = \frac{b}{a}$; and when $b = 0$, the axis of x is parallel to one of the axes of the conic section; hence if $m_3 = -m_1$, $m_4 = -m_2$; or if two opposite sides of a quadrilateral inscribed in a conic section be equally inclined to the axis, the two remaining sides will be equally inclined to the axis.

In the same manner it appears that the diagonals will be equally inclined to the axis.

45. Let two conjugate diameters CP, CD (fig. 171) be taken for the axes of x and y respectively, then $b = 0$, and if

pq be any chord parallel to CD, and $y = m_1 x$, $y = m_3 x$ be the equations to Cp, Cq respectively; since pq is bisected by CP, $m_3 = -m_1$; hence $m_4 = -m_2$; and if Cp', Cq' be drawn through C parallel to the second and fourth sides, $p'q'$ will be bisected by CP, and therefore parallel to CD.

Hence if three sides taken in order of a quadrilateral inscribed in a conic section be parallel to Cp, Cp', Cq drawn from the centre C, join pq, and draw $p'q'$ parallel to pq; then the fourth side will be parallel to Cq'.

This will be equally true if the diagonals of the quadrilateral be taken instead of the second and fourth sides.

46. When the first and third sides coincide with AB, which is parallel to CP, then the points p, q coincide in P, and CP bisects the evanescent chord joining them; and the remaining sides of the quadrilateral will be in the directions of the tangents at A and B.

Draw Cp' parallel to the tangent at A, and $p'q'$ an ordinate to CP; then Cq' will be parallel to the remaining side of the quadrilateral or to the tangent at B.

Hence it appears that if Cp', Cq' be conjugate to CA, CB respectively; AB, $p'q'$ will be parallel to a pair of conjugate diameters.

47. If the first and third sides coincide with AB, (fig. 172) the second and fourth sides will be in the directions of the tangents at A and B, and if $m_1 = m_3 = \dfrac{b}{2a}$; $m_2 + m_4$ will be constant for all chords parallel to AB; but if PP' be drawn through C parallel to AB, since the tangents at P and P' are parallel, $m_2 = m_4$, and $m_2 + m_4$ cannot $= \dfrac{b}{a}$ unless the tangent at P is perpendicular to the axis of x; and CT is parallel to the tangent at P; hence $m_2 + m_4 = \cot ATC - \cot BTC$ is constant for all chords parallel to AB.

48. If the origin be changed to any point in the conic section, a, b, c remain unchanged, and $f = 0$; hence if m_1, m_2,

m_3 be given; m_4 and λ will be known from equations (1), (2); and $\frac{c_1 c_3}{c_2 c_4} = -\lambda$ from the equation (5); but if p_1, p_2, p_3, p_4 be the perpendiculars from the origin upon the four sides

$$p_1 = \frac{c_1}{\sqrt{1 + m_1^2}}; \quad p_2 = \frac{c_2}{\sqrt{1 + m_2^2}} \text{ \&c};$$

$$\therefore \frac{p_1 p_3}{p_2 p_4} = -\lambda \frac{\sqrt{(1 + m_2^2)(1 + m_4^2)}}{\sqrt{(1 + m_1^2)(1 + m_3^2)}};$$

and is therefore constant.

Hence if perpendiculars be drawn from any point of a conic section upon the four sides of a quadrilateral inscribed in the conic section, $p_1 p_3$ will bear an invariable ratio to $p_2 p_4$. (Senate-House Problems, Thursday, Jan. 7, 1847, 1 ... 4. Quest. 14.)

The same will also be manifestly true of any quadrilateral, three of whose sides are parallel to three fixed lines.

49. If $c_1 + c_3 = \frac{d}{a}$; then $c_2 + c_4 = \frac{d}{a}$; and when $d = 0$, the origin is the middle point of the chord which is the axis of y; and if $c_3 = -c_1$, $c_4 = -c_2$.

Hence, (1) if A (fig. 173) be the middle point of a chord of a conic section, and PQ, RS meet the chord in two points B, b equidistant from A; the two remaining sides RQ, PS will meet the chord in two points C, c equidistant from A.

In the same manner it may be shewn that PR, QS meet the chord in two points equidistant from A.

(2) If PQ, SR be any two chords passing through A the middle point of BD; (fig. 174) RQ, SP will meet the chord BD in two points C, c equidistant from A.

Similarly QS, PR will meet BD in two points C', c' equidistant from A.

(3) If PQ, SR pass through A and become coincident, SP, QR become the tangents at P and Q; hence if A be the middle point of a chord, the tangents at the extremities of any chord PQ passing through A will meet the first chord in two points equidistant from A.

(4) If the points B, D (fig. 175) be made to coincide, the line $CBAc$ will become a tangent at A; and if PQ, SR meet the tangent in two points E, e equidistant from A; PS, RQ will meet the tangent in two points C, c equidistant from A; and also PR, QS will meet the tangent in two points C', c' equidistant from A.

(5) If the points Q and R (fig. 176) coincide, then QR becomes a tangent at Q; and if PQ, QS meet the chord or tangent CAc in two points E, e equidistant from A, SP and the tangent at Q will meet the chord or tangent in two points C, c equidistant from A.

50. (1) If $c_1 c_3 = \dfrac{f}{a}$, then $c_2 c_4 = \dfrac{f}{a}$; and if A be the origin, and ABC (fig. 177) the axis of y, $AB \,.\, AC = \dfrac{f}{a}$; hence if the first and third sides PQ, RS of a quadrilateral figure inscribed in a conic section cut a straight line ABC in two points E, F such that $AE \,.\, AF = AB \,.\, AC$; then the second and fourth sides SP, QR will cut ABC in two points G, H such that

$$AG \,.\, AH = AB \,.\, AC.$$

Similarly if the diagonals SQ, PR meet the line ABC in G', H'; $AG' \,.\, AH' = AB \,.\, AC$.

(2) If the points E, F coincide in O (fig. 178) so that $AB \,.\, AC = AO^2$, and PQ, SR be drawn through O in any direction, then PS, QR will meet ABC in two points G, H such that $AG \,.\, AH = AO^2$.

(3) If RS be made to coincide with PQ (fig. 179), then PS, QR become tangents at P and Q, and will intersect ABC in two points G, H such that $AG \,.\, AH = AO^2$ is constant for all chords drawn through O.

(4) If $c_1 = 0$, then $c_3 = \infty$; the first side QP (fig. 180) passes through the origin A, and the third side RS is parallel to ABC; hence if PS, QR meet ABC in G and H,

$$AG \,.\, AH = AB \,.\, AC$$

for all positions of PQ, QR.

(5) If P and Q coincide, then AP (fig. 181) becomes a tangent meeting a chord ABC in A: PR is any line; RS parallel to ABC; then $AG \,.\, AH$ is constant for all positions of PR.

51. If $d = 0$, the origin is the middle point of a chord; and if $c_1 c_2 = -\dfrac{f}{a}$, then $c_1 + c_3 + \lambda(c_2 + c_4) = 0$;

$c_1 c_3 + \lambda c_2 c_4 = -c_1 c_2 (1 + \lambda)$; or $c_1(c_2 + c_3) + \lambda c_2 (c_1 + c_4) = 0$;

$$\therefore\ c_1(c_2 + c_3) - c_2(c_1 + c_4)\left(\frac{c_1 + c_3}{c_2 + c_4}\right) = 0;$$

hence $c_1(c_2 + c_3)(c_2 + c_4) = c_2(c_1 + c_3)(c_1 + c_4)$,

$$\text{or } c_1 c_2 (c_2 - c_1) = c_3 c_4 (c_2 - c_1); \quad \therefore\ c_3 c_4 = c_1 c_2 = -\frac{f}{a};$$

hence if A be the middle of the chord BD (fig. 182), and PQ, QR meet AB in C, C' such that $AC \,.\, AC' = AB^2$; then RS, PS will meet AB in c, c' such that $Ac \,.\, Ac' = AB^2$.

(2) If $c_1 = 0$, or the first chord PQ passes through A, (fig. 183) $c_2 = \infty$, and QR is parallel to AB; hence if RS, SP meet AB in c, c'; $Ac \,.\, Ac' = AB^2$ for all positions of SP, PQ.

(3) If PQ be a diameter passing through A (fig. 184); the points Q, R coincide; and if PS, QS be lines drawn to any point S of the curve, from the extremities of a diameter PQ, meeting an ordinate to PQ in c', c; then $Ac \,.\, Ac' = AB^2$ and is invariable for every position of S.

52. From equations (1) and (2).

If $\frac{m_1 + m_3}{m_1 m_3} = \frac{b}{c}$; then $\frac{m_2 + m_4}{m_2 m_4} = \frac{b}{c}$; but if $m_1 = 0$, $m_3 = 0$, or the first and third lines are parallel to the axis of x, then $\frac{1}{m_2} + \frac{1}{m_4} = \frac{b}{c}$; hence if PQ, RS be two parallel chords; (fig. 185) the sum of the cotangents of the angles which PS, QR, or PR, QS make with PQ *in the same direction* is constant.

(2) If PQ, RS coincide, PS, QR become tangents, or the sum of the cotangents of the angles which a pair of tangents at the extremities of any chord parallel to a fixed line makes with that line *in the same direction* is constant.

53. If $\frac{c_1 + c_3}{c_1 c_3} = \frac{d}{f}$; then $\frac{c_2 + c_4}{c_2 c_4} = \frac{d}{f}$; and since

$\frac{d}{f} = -\left(\frac{1}{AB} + \frac{1}{AC}\right)$; if $\frac{1}{c_1} + \frac{1}{c_3} = -\left(\frac{1}{AB} + \frac{1}{AC}\right)$;

then $\frac{1}{c_2} + \frac{1}{c_4}$ is constant; and will be the same whether we take the second and fourth sides or the diagonals of the quadrilateral.

(2) If $c_1 = 0$, then $c_3 = 0$, and the first and third chords PQ, RS (fig. 186) pass through the origin A; therefore PS, QR meet ABC in two points E, D such that $\frac{1}{AD} + \frac{1}{AE}$ is constant.

(3) If PQ, RS coincide, (fig. 187) PS, QR become tangents, and the tangents at the extremities of any chord PQ passing through a fixed point A will meet a given line ABC in two points T, T', such that $\frac{1}{AT} + \frac{1}{AT'}$ is invariable.

54. If $d = 0$, and $f = 0$; the axis of y becomes a tangent to the curve, and from equations (3) and (5) $\frac{1}{c_1} + \frac{1}{c_3} = \frac{1}{c_2} + \frac{1}{c_4}$;

hence if the four sides of a quadrilateral inscribed in a conic section meet a tangent at A (fig. 188) in the points P_1, P_2, P_3, P_4 respectively, and the direction AP_1 be considered positive, $\frac{1}{AP_1} + \frac{1}{AP_2} = \frac{1}{AP_3} + \frac{1}{AP_4}$; and if the diagonals meet the tangent in Q_2, Q_4,

$$\frac{1}{AP_1} - \frac{1}{AP_3} = \frac{1}{AP_4} - \frac{1}{AP_2} = \frac{1}{AQ_2} - \frac{1}{AQ_4}.$$

55. If c and $f = 0$, the equation to the conic section becomes $ay^2 + bxy + dy + ex = 0$; which is that of a hyperbola having one of its asymptotes parallel to the axis of x; and $\frac{c_1}{m_1} \cdot \frac{c_3}{m_3} = \frac{c_2}{m_2} \cdot \frac{c_4}{m_4}$; or if A be the origin in the curve; and the four sides of a quadrilateral figure meet a line passing through A and parallel to one of the asymptotes in P_1, P_2, P_3, P_4; then $AP_1 \cdot AP_3 = AP_2 \cdot AP_4$; and if the diagonals of the quadrilateral meet the same line in Q_2, Q_4;

$$AQ_2 \cdot AQ_4 = AP_2 \cdot AP_4.$$

56. If $a = 0$, then $\lambda = -1$, and the curve is a hyperbola having the axis of y parallel to one of the asymptotes; and if a, and $c = 0$, $m_1 m_3 - m_2 m_4 = 0$; in this case the axes are parallel to the two asymptotes.

Hence if a quadrilateral figure be inscribed in a rectangular hyperbola, the product of the tangents of the angles which the first and third sides make with either of its asymptotes, is equal to the product of the tangents of the angles made by the remaining sides with the same asymptote.

57. If a and $d = 0$, the axis of y becomes one of the asymptotes, and $c_1 + c_3 = c_2 + c_4$; hence if four sides of a quadrilateral figure inscribed in a hyperbola meet one of its asymptotes in the points P_1, P_2, P_3, P_4;

$$AP_1 + AP_3 = AP_2 + AP_4; \quad \therefore\ P_1P_2 = P_3P_4;$$

also $P_1P_4 = P_2P_3$; or the two adjacent sides intercept the same portion of the asymptotes as the remaining two.

If the diagonals of the quadrilateral, meet the asymptote in Q_2, Q_4; then $AP_1 + AP_3 = AQ_2 + AQ_4$: or $P_1Q_2 = P_3Q_4$; similarly $P_2Q_2 = P_4Q_4$.

58. If a and $f = 0$, $c_1c_3 = c_2c_4$. In this case the axis of y is in the curve and parallel to one of the asymptotes; and if the four sides and the two diagonals meet the axis of y in P_1, P_2, P_3, P_4, and Q_2, Q_4 respectively; then

$$AP_1 \,.\, AP_3 = AP_2 \,.\, AP_4 = AQ_2 \,.\, AQ_4;$$

and when P_1, P_3 are fixed points in the axis of y, $AP_2 \,.\, AP_4$, and $AQ_2 \,.\, AQ_4$ are invariable.

59. If b and $c = 0$, the curve becomes a parabola with the axis of x parallel to the axis of the parabola; and from equations (1), (2) we have $\dfrac{1}{m_1} + \dfrac{1}{m_3} = \dfrac{1}{m_2} + \dfrac{1}{m_4}$. Hence the sums of the cotangents of the angles which each pair of opposite sides, and the diagonals of a quadrilateral figure inscribed in a parabola make with its axis *in the same direction* are respectively equal.

60. Since the form of equations (1), (2) which connect m_1, m_2, m_3, m_4 is similar to the form of equations (3), (5) which connect c_1, c_2, c_3, c_4; and the quantities are similarly involved in equation (4); if any relation be found between m_1, m_2, m_3, m_4; a corresponding relation may be determined between c_1, c_2, c_3, c_4; and vice versâ.

The following example will be sufficient to illustrate the preceding remark.

If two tangents be drawn to a conic section the tangents of whose inclinations to the axis m_1, m_3 are such that m_1m_3, or $m_1 + m_3$, or $\dfrac{1}{m_1} + \dfrac{1}{m_3}$ is constant, the locus of the intersection of the tangents will in each case be a conic section.

From this we may infer that if a pair of tangents meet any diameter at distances c_1, c_3 from the centre such that c_1c_3,

or $c_1 + c_3$, or $\frac{1}{c_1} + \frac{1}{c_3}$ is constant, the locus of the intersection of the tangents will in each case be a conic section.

61. If *AMN*, *APQ* (fig. 189) be any two chords drawn through a fixed point *A* to a conic section, the straight lines joining the points of intersection of *MP*, *NQ*, and of *MQ*, *NP* will be the chord joining the points of contact of two tangents drawn from *A*.

Let *A* be taken for the origin; and let the equations to *AMN*, *MP*, *APQ*, *QN* respectively be $y + m_1 x = 0$;

$$y + m_2 x + c_2 = 0; \quad y + m_3 x = 0; \quad y + m_4 x + c_4 = 0;$$

hence the equation to the conic section is

$$(y + m_1 x)(y + m_3 x) + \lambda(y + m_2 x + c_2)(y + m_4 x + c_4) = 0; \quad (1)$$

and in order that this may be identical with the given equation to the conic section $ay^2 + bxy + cx^2 + dy + ex + f = 0$; we must have $a\lambda(c_2 + c_4) = d(1 + \lambda)$;

$$a\lambda(c_2 m_4 + c_4 m_2) = e(1 + \lambda); \quad a\lambda c_2 c_4 = f(1 + \lambda);$$

$$\therefore \frac{1}{c_2} + \frac{1}{c_4} = \frac{d}{f}; \quad \frac{m_2}{c_2} + \frac{m_4}{c_4} = \frac{e}{f};$$

but for the intersection of $y + m_2 x + c_2 = 0$; $y + m_4 x + c_4 = 0$, we have

$$\left(\frac{1}{c_2} + \frac{1}{c_4}\right) y + \left(\frac{m_2}{c_2} + \frac{m_4}{c_4}\right) x + 2 = 0, \text{ or } \frac{d}{f} y + \frac{e}{f} x + 2 = 0;$$

hence the second and fourth sides intersect in the straight line $dy + ex + 2f = 0$, which is independent of m_1, m_3; and since equation (1) will equally represent the equation to the conic section when $y + m_2 x + c_2 = 0$, $y + m_4 x + c_4 = 0$ are the equations to *NP*, *MQ*; the locus of *S* the intersection of *MQ*, *NP* is the straight line $dy + ex + 2f = 0$; and the locus *R* of the intersection of *MP*, *NQ* is the same straight line; therefore

$dy + ex + 2f = 0$ is the equation to the straight line joining R and S.

Since RS is independent of the position of AMN, and APQ, let P and Q move up to T, and M and N to T', so that AT, AT' are tangents drawn from A; then MP, NQ become coincident with the chord of contact TT'; and since they always meet in the same straight line, RS must be the chord of contact of two tangents drawn from A.

The equation to the chord of contact of two tangents drawn from A is $dy + ex + 2f = 0$.

62. Since RST' will be the same straight line in whatever direction the lines AMN, APQ be drawn; let $AP'Q'$ be drawn very near APQ; then PP', QQ' are ultimately tangents at P and Q; but PP', QQ' will intersect in the chord of contact RS; hence a pair of tangents at the extremities of PQ will meet in the chord of contact of two tangents drawn from A.

63. If a pair of tangents be drawn at the extremities of two chords PQ, MN passing through A, the straight line joining the points of intersection of each pair will be the chord of contact of two tangents drawn from A.

64. When tangents are drawn from A, the chord of contact is RS passing through S; similarly when tangents are drawn from R the chord of contact is AS passing through S; but when pairs of tangents are drawn from any point in AR, the chords of contact will all pass through the same point; hence the chords of contact will all pass through S.

Conversely if pairs of tangents be drawn at the extremities of any chord passing through S, they will intersect in the line AR.

65. The line joining the points of intersection of pairs of tangents at the extremities of the chords MQ, NP is AR; hence MP, NQ, and also MN, PQ meet in the line joining the points of intersection of pairs of tangents at the extremities of the chords MQ, NP.

66. To draw a pair of tangents to a conic section from a given point A without it; and also to find the chord of contact.

Through A draw any two chords AMN, APQ; join MP, NQ intersecting in R; join MQ, NP intersecting in S; then RS is the chord of contact of a pair of tangents drawn from A; and if RS meet the conic section in the points T, T'; AT, AT' will be the tangents required.

67. To draw a tangent at a point A in a conic section.

Through A draw any line $CABD$ to cut the curve (fig. 190); find the chords of contact PQ, RS of pairs of tangents drawn from two points C, D without the conic section; let QP, SR intersect in T; join TA, this will be the tangent required.

Since the chords of contact of pairs of tangents drawn from any point in CD pass through the same point, the point T is the intersection of all chords of contact; and when tangents are drawn from points indefinitely near A and B, the chords of contact approach to the tangents at A and B; and since the chords of contact always pass through T, the tangents at A and B pass through T; hence TA is a tangent at the point A. See also Art. 89.

It may be observed that the tangents have been drawn by simply joining points; so that the compasses are not required.

67. (*a*) Let A, B, C, D, E (fig. 213) be five points in a conic section; produce BA, CD to meet in H; AD, BC to meet in K; and let AC, BD meet in L; join HK, HL, KL; then the tangents at A and C meet in HK. (Art. 64.)

Similarly if EC, AD meet in K', and CD, EA meet in H'; the tangents at A and C meet in $H'K'$.

Hence if HK, $H'K'$ intersect in T, TA, TC will be tangents at the points A and C.

Again, the tangents at A and D meet in HL; and the tangents at A and B meet in KL; hence if TA meet in HL,

KL in T', T''; $T'A$, $T'D$ will be tangents at A and D; and $T''A$, $T''B$ tangents at A and B: and four tangents TA, $T''B$, TC, $T'D$ have been drawn at the points A, B, C, D of the conic section which passes through the five points A, B, C, D, E.

68. If RST' be the chord of contact of a pair of tangents drawn from a point A without a conic section (fig. 189); AMN any chord passing through A, S any point in RT'; join NSP; then AP, MS will intersect in a point Q in the conic section, and the locus of Q will be the same conic section.

69. If M (fig. 191) be the vertex of a parabola whose axis is MB; and $AM = MB$; draw $BSTR$ perpendicular to MB; then BR is the chord of contact of two tangents drawn from A; and since AM does not meet the curve again, SP must be drawn parallel to AM; and the locus of the intersection of AP, MS will be the parabola MP.

70. If PAC, PDB (fig. 192) be drawn to the extremities of the axis AB of a conic section from any point P in a straight line PE perpendicular to the axis, DC will meet AB in a fixed point F.

Let DA, BC meet in G; then GF is the chord of contact of a pair of tangents drawn from P; and the chords of contact of pairs of tangents drawn from any point in PE will pass through a fixed point in the axis AB; hence the point F is fixed.

(2) If a pair of tangents be drawn from G, PF will be the chord of contact, and since it passes through F, G is a point in PE; hence DA, BC intersect in PE.

(3) If BC be produced to meet PE in G, and DG be joined it will pass through A.

71. If the curve AD is a parabola, we must suppose B removed to an infinite distance; in which case PD, GC become parallel to the axis; hence if from any point P in a line PE

perpendicular to the axis of a parabola, PAC be drawn through the vertex, and PD parallel to the axis; DC will meet the axis in a fixed point F. In this case $AF = AE$.

(2) If CG be drawn parallel to the axis, DG will pass through A.

72. If A be removed to an infinite distance PQ, MN (fig. 189) become two parallel chords; and MP, NQ, as well as MQ, NP will always intersect in a straight line, which will be a diameter to the chords.

73. If AMN, APQ (fig. 189) be any two lines drawn through A to meet a surface of the second order in M, N, and P, Q respectively; the intersection R of MP, NQ will be in the plane of contact of the enveloping cone whose vertex is A; and the locus of R will be the plane of contact.

For the section of the surface made by the plane passing through APQ, AMN will be a conic section, and the locus of R will be the line of contact of two tangents drawn to the conic section from A; and this will be a straight line on the plane of contact of the enveloping cone whose vertex is A; and the same will be true when the lines AMN, APQ are drawn through A in any other direction; hence the locus of R is the plane of contact.

In like manner it may be proved that the locus of S the point of intersection of MQ, NP is the plane of contact of the enveloping cone whose vertex is A.

74. If a hexagon be inscribed in a conic section, the points of intersection of the opposite sides will all lie in the same straight line. (Pascal's hexagram).

Let the sides of the hexagon $ABCDEF$ (fig. 194) taken in order be represented by α, β, γ, δ, ϵ, ζ; and let their equations be respectively

$$u_\alpha = 0,\ u_\beta = 0,\ u_\gamma = 0,\ u_\delta = 0,\ u_\epsilon = 0,\ u_\zeta = 0;$$

also let $u_1 = 0$ be the equation to the diagonal BE; and $U=0$ the equation to the conic section; hence

$$u_1 u_\zeta + \lambda_1 u_\alpha u_\epsilon = m_1 U;$$

$$u_1 u_\gamma + \lambda_2 u_\beta u_\delta = m_2 U;$$

$$\therefore (m_1 u_\gamma - m_2 u_\zeta) U = \lambda_1 u_\alpha u_\gamma u_\epsilon - \lambda_2 u_\beta u_\delta u_\zeta. \qquad (1)$$

If u_α and $u_\delta = 0$, the values of x and y will be the co-ordinates of the point of intersection of the sides α and δ; and from equation (1), $(m_1 u_\gamma - m_2 u_\zeta) U = 0$; but U cannot $= 0$, since the intersection of α and δ is not a point in the conic section; $\therefore m_1 u_\gamma - m_2 u_\zeta = 0$; or α and δ intersect in the straight line whose equation is $m_1 u_\gamma - m_2 u_\zeta = 0$.

Similarly, if in equation (1) we put u_β and $u_\epsilon = 0$, the lines β and ϵ intersect in the straight line $m_1 u_\gamma - m_2 u_\zeta = 0$.

Now $m_1 u_\gamma - m_2 u_\zeta = 0$ is a straight line passing through γ, ζ; hence the intersections of α, δ; β, ϵ; γ, ζ lie in the same straight line whose equation is $m_1 u_\gamma - m_2 u_\zeta = 0$.

(a) If the six chords of the conic section be taken in the order AB, BE, ED, DC, CF, FA, it will appear that the intersections of AB, CD; BE, CF; ED, FA are in the same straight line. Hence BE, CF intersect in cf.

Similarly AD, BE intersect in eb; and AD, CF in ad.

In like manner by varying the order of the points A, B, C, D, E, F, we may find several series consisting of three different points which lie in the same straight line.

75. The three diagonals of a hexagon circumscribing a conic section meet in the same point. (Brianchon's theorem).

Let $ABCDEF$ (fig. 170) be the hexagon; a, b, c, d, e, f the points of contact; then (Art. 63) AD is the chord of contact of tangents drawn from a' the intersection of dc, af; similarly BE, CF are the chords of contact of tangents drawn from a'', a''' the intersections of de, ab; and ef, cb respectively;

and a', a'', a''' lie in the same straight line (Art. 74); therefore AD, BE, CF meet in the same point.

76. Let $ABCD$ (fig. 195) be a quadrilateral figure inscribed in a conic section, this may be considered as a hexagon whose sides at the points A, C vanish, and are in the directions of the tangents at A and C; then since the opposite sides of the hexagon intersect in a straight line, if AB, DC be produced to meet in E, and AD, BC to meet in F, the two remaining sides of the hexagon, viz. the tangents at A and C will intersect in the line EF.

Similarly the tangents at B, D intersect in the line EF.

If the lines inscribed in the conic section be taken in the order $ABDCA$, and the tangents at A and D be considered as two evanescent chords passing through A and D; then the intersections of AB, DC; BD, AC; and the tangents at A and D will meet in a point; or the tangents at A and D will meet in EG.

Similarly, the tangents at B, C meet in EG.

These properties have been already proved (Art. 63).

77. Let $ABCD$ (fig. 195) be a quadrilateral figure circumscribing a conic section, and touching it in the points a, b, c, d;

(1) The quadrilateral figure may be considered as a hexagon whose angular points are A, a, B, C, c, D; in which case the three diagonals AC, BD, ac meet in a point.

(2) Let A, B, b, C, D, d be considered the angular points; then the three diagonals AC, BD, bd meet in a point.

Hence ac, bd pass through the intersection of AC, BD.

(3) Let A, a, B, b, C, D be considered the angular points; then Ab, Ca, BD meet in a point.

Similarly, Ac, BD, Cd meet in the same point.

And in the same manner it may be proved that Bc, Db; and also Bd, Da intersect in AC.

78. If a triangle ABC (fig. 196) be described about a conic section, and touch it in the points a, b, c, since ABC may be considered as a hexagon whose angular points are A, c, B, a, C, b; the diagonals Aa, Bb, Cc intersect in a point.

Hence if a conic section inscribed in a triangle touches two sides BC, AC in the points a, b; join Aa, Bb to meet in D; and produce CD to meet AB in c; c will be the point of contact of the side AB.

79. Let a, b, c, d (fig. 195) be the points of contact of a quadrilateral figure $ABCD$ circumscribing a conic section; produce AB, DC to meet in E; AD, BC to meet in F; also let AC, BD meet EF in H and K respectively; and let AC, BD intersect in G.

(1) ab, dc, (Art. 63) intersect in the line joining the points C, A of intersection of the pairs of tangents at the extremities of the chords bc, ad; hence ab, dc intersect in AC.

(2) ab, dc intersect in the line EF joining the points E, F of the intersection of the pairs of tangents at the extremities of the chords ac, db (Art. 65); hence ab, dc intersect in H.

Similarly ad, bc intersect in K.

(3) ac, bd (Art. 64) will meet in the line AC joining the points of intersection A, C of the pairs of tangents at the extremities of the chords ad, bc.

Similarly ac, bd meet in BD; hence ac, bd will meet in G, the point of intersection of AC, BD.

80. A conic section is to be inscribed in a quadrilateral figure $ABCD$ (fig. 195) so as to touch the side AB in a; to find the points b, c, d of contact with the three remaining sides.

Join AC, BD meeting in G; let AB, DC meet in E; and AD, BC in F; produce AC to meet EF in H and aG to meet DC in c; join Ha, Hc meeting BC, AD respectively in b, d; and the points b, c, d are determined.

81. When a pentagon is described about a conic section to find the five points of contact with the sides.

Let $ABCDE$ (fig. 197) be the pentagon, and a the point of contact with the side AB which it is proposed to determine; then Aa, aB, BC, CD, DE, EA may be considered as the six sides of a circumscribing hexagon; and the three diagonals Da, AC, BE meet in a point. Hence join BE, AC, meeting in G; draw DG meeting AB in a; then a is the point of contact with the side AB. In like manner the points of contact with the remaining sides may be determined.

82. Having given five points A, B, C, D, E of a conic section, to determine a sixth point in any given direction DF passing through one of the given points.

Let F (fig. 198) be the sixth point which is to be determined; then $ABCDFE$ is a hexagon inscribed in a conic section; and the opposite sides BA, DF; AE, CD; and BC, FE meet in three points K, K', K'' which are in the same straight line. Hence produce BA, DF to meet in K; CD, EA to meet in K', and let CB meet $K'K$ in K''; draw $K''E$ meeting DF in F; F is a point in the conic section. In like manner a point of the conic section may be determined in any other direction passing through D; and the conic section can be traced by a consecutive series of points.

83. To determine geometrically the centre of a conic section which passes through five given points.

Through any given point as D, draw (Art. 82) a chord DF in a direction parallel to BC; bisect DF, BC: the line joining the middle points will pass through the centre. Similarly, find a chord DG parallel to EA; the straight line joining the middle points of DG, EA will pass through the centre, and the intersection of the two diameters will be the centre required.

Cor. If the two diameters are parallel, the centre is at an infinite distance and the curve is a parabola.

84. To find geometrically a consecutive series of points in a conic section which shall touch five given straight lines.

Find (Art. 81) the five points of contact A, B, C, D, E with the five given straight lines; and determine (Art. 82) a consecutive series of points in the conic section which shall pass through the five points A, B, C, D, E. The centre may also be determined by Art. 83, or Art. 31.

85. Find a consecutive series of points in a conic section which shall pass through three given points, and touch a straight line in a given point, (fig. 199.)

Let A be the given point in the given straight line AK'; B, C, D the three given points; BE the direction of a chord passing through B; E a point in the conic section, which it is proposed to determine; then $ABECD$ may be considered a hexagon inscribed in the conic section, whose sides are the evanescent chord at A in the direction of the tangent AK', AB, BE, EC, CD, DA; and since the opposite sides meet in a straight line, if AB, DC meet in K, and BE, DA in K'', then the tangent at A and CE will meet in K' a point in KK''; hence produce AB, DC to meet in K, DA, BE to meet in K'', and let the tangent AK' meet KK'' in K'; join $K'C$, it will cut BE in the required point E. And in like manner we may find a point in the conic section in any other direction drawn from B.

86. Find a consecutive series of points in a conic section which shall touch four straight lines, and one of them in a given point.

Find (Art. 80) the three points of contact B, C, D of the remaining sides; and determine a series of points in the conic section which shall pass through B, C, D; and touch the first side in the point A. (Art. 85).

87. Find a consecutive series of points which shall touch three given straight lines AB, AC, BC; and two of them BC, CA in given points a, b, (fig. 200.)

Find (Art. 78) the point of contact c with the side AB; and let cd, be the direction of a chord of the conic section

which passes through c; produce ba, cd to meet in E; join EB and let bc meet EB in F; then if d be a point in the conic section, $cdab$ may be considered as a hexagon inscribed in the conic section, whose sides are the evanescent chord at c in the direction cB; cd, da, the evanescent chord at a in the direction Ba, ab, bc; and the points of intersection of the opposite sides are in the same straight line; hence bc, ad meet in EB; therefore produce bc to meet EB in F; join Fa, which will intersect cd in the point required.

88. Find a series of points in a conic section which shall pass through a given point C; and touch two given straight lines OA, OB in given points A, B, (fig. 201).

Let AD be the direction of a chord of the conic section through A; D the extremity of the chord which is to be determined; then $ADCB$ may be considered a hexagon inscribed in the conic section whose sides are the tangent at A, AD, DC, CB, the tangent at B, and BA; and the opposite sides meet in the same straight line. Let CB meet OA in K, and AD meet OB in K'; then DC will meet AB in the line KK'; hence produce AB to meet KK' in K'', join $K''C$ meeting AD in D; then D will be a point in the conic section.

89. Having given five points A, B, C, D, E, to draw a tangent at A to the conic section which passes through them, (fig. 202).

Since $ABCDE$ may be considered as a hexagon inscribed in a conic section whose sides are AB, BC, CD, DE, EA and the tangent at A; K, K', K'' (the points of intersection of AB, DE; BC, EA, and of DC and the tangent at A) lie in the same straight line; hence join KK', and let DC meet KK' in K'', $K''A$ will be the tangent required.

The construction here given is more simple than that in Art. 67.

90. When a conic section passes through five given points

A, B, C, D, E; find the extremities of the diameter to which a chord AF drawn from A in a given direction is an ordinate.

Find F (fig. 203) the extremity of the chord AF (Art. 82); and O the centre of the conic section (Art. 83); bisect AF in G; draw a tangent AT at the point A (Art. 89); and draw OG meeting AT in T; then OGT is the direction of the diameter to AF; and if $OD' = Od'$ be taken a mean proportional between OG and OT, D', d' will be the extremities of the diameter.

91. Find the extremities of the diameter which is parallel to AF, or conjugate to $D'd'$.

Through the centre O draw OK parallel to AF; and from E one of the given points, draw $D'E$, $d'E$ meeting OK in H, K; take $OL = Ol$ a mean proportional between OH and OK; then (Art. 51, 3) L and l will be the extremities of the diameter.

92. Find the extremities of a chord GH drawn in any given direction.

Through the given point A, (fig. 204) draw the chord AF parallel to GH (Art. 82); draw the diameter $D'd'$ to which AF is an ordinate (Art. 90), and let it meet GH in M; take E one of the given points, and join $D'E$, $d'E$ meeting GH in K, L; take $MH = MG$ a mean proportional between MK and ML; then (Art. 51, 3) G, H are the extremities required of the chord GH.

93. Having given two semi-conjugate diameters CP, CD of a conic section, to determine the magnitude and position of the axes. (fig. 205).

Let aA be the axis-major; CP, CD the given semi-conjugate diameters; draw PT parallel to CD which is therefore a tangent at P;

let $CP = a'$, $CD = b'$, $\angle ACP = \theta$, $\angle CPT = \angle PCD' = \alpha$;

$$\text{then } b' \cos(\alpha - \theta) = \frac{a}{b} a' \sin\theta;$$

$$b' \sin(\alpha - \theta) = \frac{b}{a} a' \cos\theta;$$

$$\therefore\ b'^2 \sin 2(\alpha - \theta) = a'^2 \sin 2\theta;$$

draw PE making $\angle EPT' = \angle CPT$; and make $PE = \frac{b'^2}{a'}$; join CE; then $\angle CPE = \pi - 2\alpha$; and if $\angle PCE = \phi$; $\angle PEC = 2\alpha - \phi$;

$$\therefore\ \frac{PE}{PC} = \frac{b'^2}{a'^2} = \frac{\sin\phi}{\sin(2\alpha - \phi)} = \frac{\sin 2\theta}{\sin(2\alpha - 2\theta)};\quad \therefore\ \phi = 2\theta;$$

or CA bisects angle PCE.

Hence make $\angle T'PE = \angle CPT$; and $PE = \frac{CP^2}{CD}$; join CE, and bisect $\angle PCE$, this will give the direction of the axis-major.

The axis-major CA always falls within the acute angle PCD'; for $\tan\theta \,.\, \tan(\alpha - \theta) = \frac{b^2}{a^2}$;

$$\therefore\ \tan(\alpha - \theta) = \frac{b^2}{a^2} \tan\left(\frac{\pi}{2} - \theta\right) < \tan\left(\frac{\pi}{2} - \theta\right);$$

$$\text{hence } \alpha < \frac{\pi}{2};\ \text{ since } \theta \text{ may be taken } < \frac{\pi}{2}.$$

Draw CB, PM perpendicular to CA, and PN perpendicular to CB; then if the tangent at P meets CA, CB in T', T; CA, CB are mean proportionals between CM, CT'; and CN, CT.

To find the foci, take BS, $BH = CA$; and S, H will be the foci required.

94. Find the magnitude and position of the axes of the conic section which passes through five given points.

Find a pair of conjugate diameters $D'd'$, Ll (fig. 203) (Arts 90, 91); and thence Art. 93, the axes and the foci of the conic section may be determined.

Cor. When the curve is a parabola, the vertex and focus may be found by Art. 37. (γ)

95. Having given five points, draw a tangent to the conic section which passes through them, from a given point without it.

Let H (fig. 206) be the given point without the conic section; A, B two of the given points; join HA, HB; and find F, G the extremities of the chords (Art. 82); join BA, GF meeting in K; and AG, BF meeting in L; then KL is the chord of contact of a pair of tangents drawn from H (Art. 61); find P, Q the extremities of this chord (Art. 92); and HP, HQ will be the tangents required.

96. In like manner we may find the extremities of any chord, and draw a tangent from a given point either in or without any of the conic sections described in Arts. 84—88; and also find the magnitude and position of the axes, since in each case we are enabled to determine five points of the conic section.

97. To inscribe geometrically in a given conic section a triangle whose three sides shall pass through three given points A, B, C.

If from any point P in the conic section Pa_1, Pb_1 (fig. 207) be drawn through A, B; then the straight line a_1b_1 will always touch a conic section. (Appendix I. Art. 5).

Let a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, be any five positions of this line; and from the given point C draw two tangents CED, $CD'E'$ to the conic section which touches them (Arts. 95, 96); let these tangents meet the given conic section in the chords DE, $D'E'$ respectively, join DAF, EBF; $D'AF'$, $E'BF'$; then DEF, $D'E'F'$ will be the two triangles inscribed in the conic section whose sides pass through the three given points A, B, C.

98. In a conic section which passes through five given points, it is required to inscribe a triangle whose sides shall pass through three given points A, B, C.

Let P be one of the given points; through A, B, draw

the chords Pa_1, Pb_1 (Art. 82); join $a_1 b_1$. Similarly, we can determine $a_2 b_2$, $a_3 b_3$, $a_4 b_4$, $a_5 b_5$ by taking the four remaining points instead of P; from C as before draw the two tangents CED, $C'D'E'$ to the conic section which touches the five lines

$$a_1 b_1,\ a_2 b_2,\ a_3 b_3,\ a_4 b_4,\ a_5 b_5\,;$$

find D, E, D', E' the extremities of the chords CED, $C'D'E'$ (Art. 92); join DA, EB meeting in F; and $D'A$, $E'B$ meeting in F'; DEF, $D'E'F'$ will be the two triangles required.

99. In like manner we may inscribe geometrically a triangle whose sides shall pass through three given points, in any of the conic sections enumerated in Arts. 84—88, although the conic section itself is not traced.

100. To inscribe in a given conic section a polygon whose n sides taken in order shall pass through n fixed points, A_1, $A_2 \ldots A_n$.

If $(n-1)$ sides of a polygon taken in order pass through $(n-1)$ fixed points, the n^{th} side will always touch a conic section. (Appendix I. Art. 6). From any point P in the conic section let a polygon be formed whose sides successively pass through A_1, $A_2 \ldots A_{n-1}$; and let $a_1 b_1$ be the position of the n^{th} side; in like manner let $a_2 b_2$, $a_3 b_3$, $a_4 b_4$, $a_5 b_5$ be four more positions of the n^{th} side, and from A_n draw as in Art. 97, two tangents $B_n B_{n-1}$, $B'_n B'_{n-1}$ to the conic section which touches the five lines $a_1 b_1$, $a_2 b_2$, $a_3 b_3$, $a_4 b_4$, $a_5 b_5$; these will be the n^{th} sides of the two polygons which can be inscribed in the conic section so that its sides may pass through the n given points; draw $B_{n-1} B_{n-2}$ through A_{n-1}, $B_{n-2} B_{n-3}$ through A_{n-2} &c., $B_3 B_2$ through A_3, $B_2 B_1$ through A_2; then $B_1 B_{n-1}$ will pass through A_1, and $B_1 B_2 \ldots B_n$; $B'_1 B'_2 \ldots B'_n$ will be the two polygons required.

101. Since the construction can be completed as in Art. 98, when only five points of the conic section are given, or in any of the conic sections described in Arts 84—88, we are thus enabled to inscribe in any of the above cases a polygon whose n sides taken in order shall pass through n fixed points, although the conic section itself is not actually traced.

102. To find under what limitations the problem is possible.

Since the n^{th} side which passes through A_n is a tangent to the conic section which touches a_1b_1, $a_2b_2 \ldots a_5b_5$; if A_n be without this conic section, we can draw two tangents, which will give two polygons.

If A_n is a point in the conic section, there will only be one tangent, and only one polygon can be described.

If A_n is within the conic section, it is impossible to draw a tangent from A_n, and the problem is impossible.

Hence if A be the point of contact with $a_1\, b_1$, and AA_n be drawn, find B the extremity of the chord AA_n; and there will be two polygons, one polygon or none according as A_n falls beyond B, upon B, or within B.

103. (*a*) If two chords AB, AC (fig. 208) be drawn from a point A in a curve of the second order equally inclined to a given line AD, the line joining the extremities B and C will always pass through a fixed point. (Senate House Problems, No. 15, Thursday, Jan. 7, 1847, 1...4.)

Let AD be the axis of x; then if $y - m_1x = 0$;

$$y + m_1x = 0; \quad y - m_2x = 0, \text{ and } y - m_4x - c_4 = 0,$$

be the equations to AB, AC, the tangent at A and BC respectively, the equation to the conic section is

$$(y - m_1x)(y + m_1x) + \lambda(y - m_2x)(y - m_4x - c_4) = 0;$$

or $(1+\lambda)y^2 - \lambda(m_2+m_4)xy + (\lambda m_2m_4 - m_1^2)x^2 - \lambda c_4y + \lambda m_2c_4x = 0$;

and if the equation to the conic section be

$$ay^2 + bxy + cx^2 + dy + ex = 0; \text{ we have}$$

$$\frac{\lambda c_4}{\lambda(m_2+m_4)} = \frac{d}{b}; \text{ or } c_4 = \frac{d}{b}(m_2+m_4); \text{ and } \frac{\lambda m_2c_4}{-\lambda c_4} = \frac{e}{d};$$

$$\therefore\ m_2 = -\frac{e}{d}; \quad \text{hence } c_4 = \frac{d}{b} m_4 - \frac{e}{b};$$

therefore the equation to CB becomes $y + \frac{e}{b} = m_4\left(x + \frac{d}{b}\right)$; which is the equation to a straight line passing through a fixed point whose co-ordinates are $-\frac{d}{b}$; $-\frac{e}{b}$.

Let the angles BAD, CAD be indefinitely diminished; then CB ultimately becomes a tangent at D; hence the chord CB always passes through a fixed point in the tangent at D.

(β) If AD be a chord of a surface of the second order; let any plane pass through AD, and let two chords AB, AC be drawn in the plane section (which will be a curve of the second order) equally inclined to AD; the lines joining B and C will all pass through a point in the tangent, at D to the plane section, which will be a line on the tangent plane to the surface at D.

Hence the lines similarly drawn in different plane sections will intersect in a series of points on the tangent plane to the curve surface at D.

If four lines AB, AC; AB', AC' be drawn in two different plane sections, BC, $B'C'$ are two lines in different planes; and if they meet at all they will meet in some point in the intersection AD of the two planes.

Hence all the possible intersections of the chords passing through the points B, C so drawn that AB, AC may be equally inclined to a fixed chord AB in a surface of the second order, will be a plane curve on the tangent plane to the surface at the point D, and the line AD.

104. Let AD be a fixed line, AC, AB two lines equally inclined to AD; BC the chord joining the points B, C; $B'C'$ any other position of BC; produce BC, $B'C'$ to meet in E; DE will be a tangent at the point D; which may be drawn at a given point D in a conic section.

105. If AB, AC be two chords of a conic section, to find the condition that BC may pass through a fixed point.

Let $y + m_1x = 0$; $y + m_3x = 0$; $y + m_2x + c_2 = 0$; and $y + m_4x = 0$; be the equations to AB, AC, BC and the tangent at A respectively; then the equation to the conic section is

$$(y + m_1x)(y + m_3x) + \lambda(y + m_2x + c_2)(y + m_4x) = 0;$$

and making this coincide with the equation to the conic section $ay^2 + bxy + cx^2 + dy + ex = 0$, we have

$$a\left\{(m_1 + m_3) + \lambda(m_2 + m_4)\right\} = b(1 + \lambda);$$

$$a(m_1m_3 + \lambda m_2m_4) = c(1 + \lambda);$$

$$a\lambda c_2 = d(1 + \lambda);$$

$$a\lambda c_2m_4 = e(1 + \lambda);$$

$$\therefore\ m_4 = \frac{e}{d}, \text{ and } \frac{1}{\lambda} = \left(\frac{ac_2}{d} - 1\right);$$

$$\text{or } m_1m_3\left(\frac{ac_2}{d} - 1\right) + \frac{em_2}{d} = \frac{c}{a}\left(1 + \frac{1}{\lambda}\right) = \frac{cc_2}{d}; \qquad (1)$$

$$(m_1 + m_3)\left(\frac{ac_2}{d} - 1\right) + m_2 + \frac{e}{d} = \frac{bc_2}{d}. \qquad (2)$$

Hence if m_1m_3, or $m_1 + m_3$ be constant; we have an equation of the form $\beta + m_2\alpha + c_2 = 0$, where β, α are constant; and the line $y + m_2x + c_2 = 0$ passes through a fixed point α, β.

Let $\beta + m_2\alpha + c_2 = 0$, be the general equation connecting m_2 and c_2;

$$\therefore\ m_1m_3\left(\frac{ac_2}{d} - 1\right) - \frac{e}{d}\left(\frac{c_2 + \beta}{\alpha}\right) = \frac{cc_2}{d};$$

$$(m + m_3)\left(\frac{ac_2}{d} - 1\right) + \frac{e}{d} - \frac{c_2 + \beta}{\alpha} = \frac{bc_2}{d};$$

and eliminating c_2, an equation of the form

$$m_1 m_3 - A(m_1 + m_3) + B = 0,$$

is determined where A, B are constant.

If this relation holds, the line BC will pass through a fixed point;

$$\therefore\ (m_1 - A)(m_3 - A) \text{ is constant,}$$

$$\text{or } \left(\frac{1}{m_1} - \frac{A}{B}\right)\left(\frac{1}{m_3} - \frac{A}{B}\right) \text{ is constant.}$$

If $B = 0$, $\dfrac{1}{m_1} + \dfrac{1}{m_3}$ is constant.

106. If the angle BAC be a right angle, $m_1 m_3 = -1$; and BC will pass through a fixed point.

To find the position of the fixed point; let C approach to A, then AC becomes a tangent, and AB a normal; hence CB becomes a normal, and the fixed point lies in the normal.

107. To draw a normal to a conic section from a given point in the curve.

Let A be the given point (fig. 209); draw any two chords AB, AB'; and AC, AC' at right angles to AB, AB' respectively; join BC, $B'C'$ meeting each other in O; then AO will be a normal at the point A. (Art. 106.)

APPENDIX III.

1. If $C'L'$ (fig. 214) be a given chord in a circle, it is required to draw through a given point G a chord CGL equal to $C'L'$.

Let O be the centre of the circle; draw OH perpendicular to $C'L'$; with centre O and radius OH describe a circle; from G draw $CGML$ touching this circle; then since CL and $C'L'$ are equally distant from the centre, $CL = C'L'$.

There will be two positions of $CGML$ corresponding to the two tangents through G.

2. If FQ, FR (fig. 214) be two fixed lines; and from any point K' in a circle $C'K'L'$, $K'C'$, $K'L'$ be drawn parallel to FR, FQ respectively; then $C'L'$ will be constant for all positions of K'. For $\angle C'K'L' = \angle QFR$ and is constant; hence $C'L'$ is constant.

3. In a given circle, it is required to inscribe a triangle so that its three sides may pass through three fixed points P, Q and R.

Let ABC (fig. 215) be the triangle required, whose three sides AB, BC, CA pass through the three fixed points P, Q, R respectively; join RP; produce it to F so that

$$FP \cdot PR = AP \cdot PB;$$

join FQ, and make $QG \cdot QF = QC \cdot QB$; produce BF to K; join CK, KL; then since a circle may be described about B, F, A and R,

$$\angle ARP = \angle ABF = \angle ABK = \angle ACK;$$

hence KC is parallel to FR.

Again, since a circle may be described about the points C, G, F, B,

$$\angle BCG + \angle BFG = 2 \text{ right angles} = \angle KFQ + \angle BFG;$$

$$\therefore \angle KFQ = \angle BCG = \angle BCL = \angle BKL;$$

hence KL is parallel to FQ; but when K' is any point in the circle and $K'C'$, $K'L'$ (fig. 214) are drawn parallel to FR, FQ, $C'L'$ will be constant (Art. 2); hence if through G the line CGL be drawn equal to $C'L'$ (Art. 1), the point C will be one of the angular points of the triangle; through Q draw QCB, and through P draw BPA; join AC, then ABC will be the triangle required.

The two positions of CGL will give two triangles which satisfy the conditions of the problem.

If OG be less than OH, the point G falls within the circle whose radius is OH; and the problem is impossible.

4. To inscribe geometrically a triangle in an ellipse whose three sides shall pass through three fixed points.

This problem is reduced to that of inscribing a triangle in the circle described upon the axis-major. (App. I. Art. 3.)

5. To inscribe geometrically a triangle, in a hyperbola, whose sides shall pass through three fixed points.

From Appendix I, Art. 4. we have,

$$(a + a_3)\, t_1 t_2 - \frac{a}{b} b_3 (t_1 + t_2) + a_3 - a = 0.$$

Let $\alpha_3 = \dfrac{a^2}{a_3}$; $\beta_3 = \dfrac{b_3}{b}\alpha_3$;

then $(a + \alpha_3)\, t_1 t_2 - \beta_3 (t_1 + t_2) + (a - \alpha_3) = 0.$ (1)

Similarly, if $\alpha_2 = \dfrac{a^2}{a_2}$; $\beta_2 = \dfrac{b_2}{b}\alpha_2$; $\alpha_1 = \dfrac{a^2}{a_1}$; $\beta_1 = \dfrac{b_1}{b}\alpha_1$;

we have $(a + \alpha_2)\, t_1 t_3 - \beta_2 (t_1 + t_3) + (a - \alpha_2) = 0$; (2)

$(a + \alpha_1)\, t_2 t_3 - \beta_1 (t_2 + t_3) + (a - \alpha_1) = 0$; (3)

which are the equations we should obtain for determining the angular points of a triangle inscribed in a circle whose radius is (a) so that the three sides may pass through three points whose co-ordinates measured from the centre are

$$\frac{a^2}{a_3}, \quad \frac{b_3}{b}\,\frac{a^2}{a_3}; \qquad \frac{a^2}{a_2}, \quad \frac{b_2}{b}\,\frac{a^2}{a_2}; \qquad \frac{a^2}{a_1}, \quad \frac{b_1}{b}\,\frac{a^2}{a_1};$$

hence if CA (fig. 146) be the semi-transverse axis; P an angular point of the triangle inscribed in the circle APD so that its sides may pass through the three points above determined; then $\angle ACP = \theta_1$; and $CR = a \sec\theta_1 =$ the abscissa of the corresponding angle of the triangle inscribed in the hyperbola so that its three sides may pass through the three points a_3, b_3; a_2, b_2; a_1, b_1.

6. To inscribe geometrically in a parabola, a triangle whose three sides shall pass through three given points.

Substitute for y_1, y_2, y_3 in equations (1), (2), (3), Art. 6, Append. I. the values $2at_1$, $2at_2$, $2at_3$ respectively; and let

$$\alpha_3 = \frac{a\,(a - a_3)}{a + a_3}, \quad \beta_3 = \frac{a b_3}{a + a_3},$$

$$\alpha_2 = \frac{a\,(a - a_2)}{a + a_2}, \quad \beta_2 = \frac{a b_2}{a + a_2},$$

$$\alpha_1 = \frac{a\,(a - a_1)}{a + a_1}, \quad \beta_1 = \frac{a b_1}{a + a_1};$$

$$\therefore \; (a + \alpha_3)\,t_1 t_2 - \beta_3\,(t_1 + t_2) + (a - \alpha_3) = 0, \text{ \&c.}$$

hence t_1, t_2, t_3 are the same as for the angular points of a triangle inscribed in a circle whose radius is a, so that its three sides may pass through three fixed points α_3, β_3; α_2, β_2; α_1, β_1, respectively.

Hence if E (fig. 15) be the vertex of a parabola; B the focus; with centre E, and radius EB describe the circle BDA, and let C be one of the angular points of the triangle

inscribed in the circle BDA, so that its three sides may pass through three points whose co-ordinates are α_3, β_3; α_2, β_2; α_1, β_1; draw BF perpendicular to AB meeting AC produced in F; then

$$\angle CEB = \theta_3 \quad \angle CAB = \frac{\theta_3}{2}, \text{ and } BF = 2at_3 = y_3;$$

or BF is equal to the ordinate of the corresponding angular point of the triangle which is inscribed in the parabola so that its sides may pass through the three given points a_3, b_3; a_2, b_2; a_1, b_1.

Hence in each of the three conic sections, the problem is reduced to that of inscribing in a given circle a triangle whose three sides shall pass through three fixed points.

7. From the equations in App. I. Art. 10, it will easily be seen (1) that when a polygon is to be inscribed in an ellipse so that its n sides may pass through n given points a_1, b_1; a_2, b_2; ... a_n, b_n; the problem will be equivalent to that of inscribing in the circle described upon the axis-major, a polygon whose n sides taken in order shall pass through the n fixed points

$$a_1, \ \frac{a}{b}b_1; \quad a_2, \ \frac{a}{b}b_2; \ \ldots \ a_n, \ \frac{a}{b}b_n.$$

(2) When the polygon is to be inscribed in a hyperbola; let a circle be described upon the transverse axis; and in this circle let a polygon be inscribed, whose n sides shall pass through the n fixed points

$$\frac{a^2}{a_1}, \ \frac{b_1}{b}\frac{a^2}{a_1}; \quad \frac{a^2}{a_2}, \ \frac{b_2}{b}\frac{a^2}{a_2}; \ \ldots \ \frac{a^2}{a_n}, \ \frac{b_n}{b}\frac{a^2}{a_n};$$

then if P be any angular point of this polygon, as in Art. 5, CR (fig. 146) will be equal to the abscissa of the corresponding angular point of the polygon which is to be inscribed in the hyperbola.

(3) When the polygon is to be inscribed in a parabola; let E (fig. 15) be the vertex; B the focus; with centre E

and radius BE describe a circle; and in this circle let a polygon be inscribed whose n sides shall pass through the n fixed points

$$\frac{a(a-a_1)}{a+a_1},\ \frac{ab_1}{a+a_1};\ \frac{a(a-a_2)}{a+a_2},\ \frac{ab_2}{a+a_2};\ \dots\ \frac{a(a-a_n)}{a+a_n},\ \frac{ab_n}{a+a_n};$$

let C be one of the angular points of this polygon; then, as in Art. 6, BF will be the value of the ordinate of the corresponding angular point of the polygon which is to be inscribed in the parabola.

8. When only five points of the conic section are given, the most simple method will be to determine the axes of the conic section, which will give geometrically in each case the circle and points through which the n sides of the subsidiary polygon are to pass; from whence the abscissæ of the corresponding angular points of the polygon inscribed in the conic section; and the angular points themselves may be determined.

For the Geometrical construction when the polygon is to be inscribed in a circle, the reader is referred to the Liverpool Apollonius by J. H. Swale.

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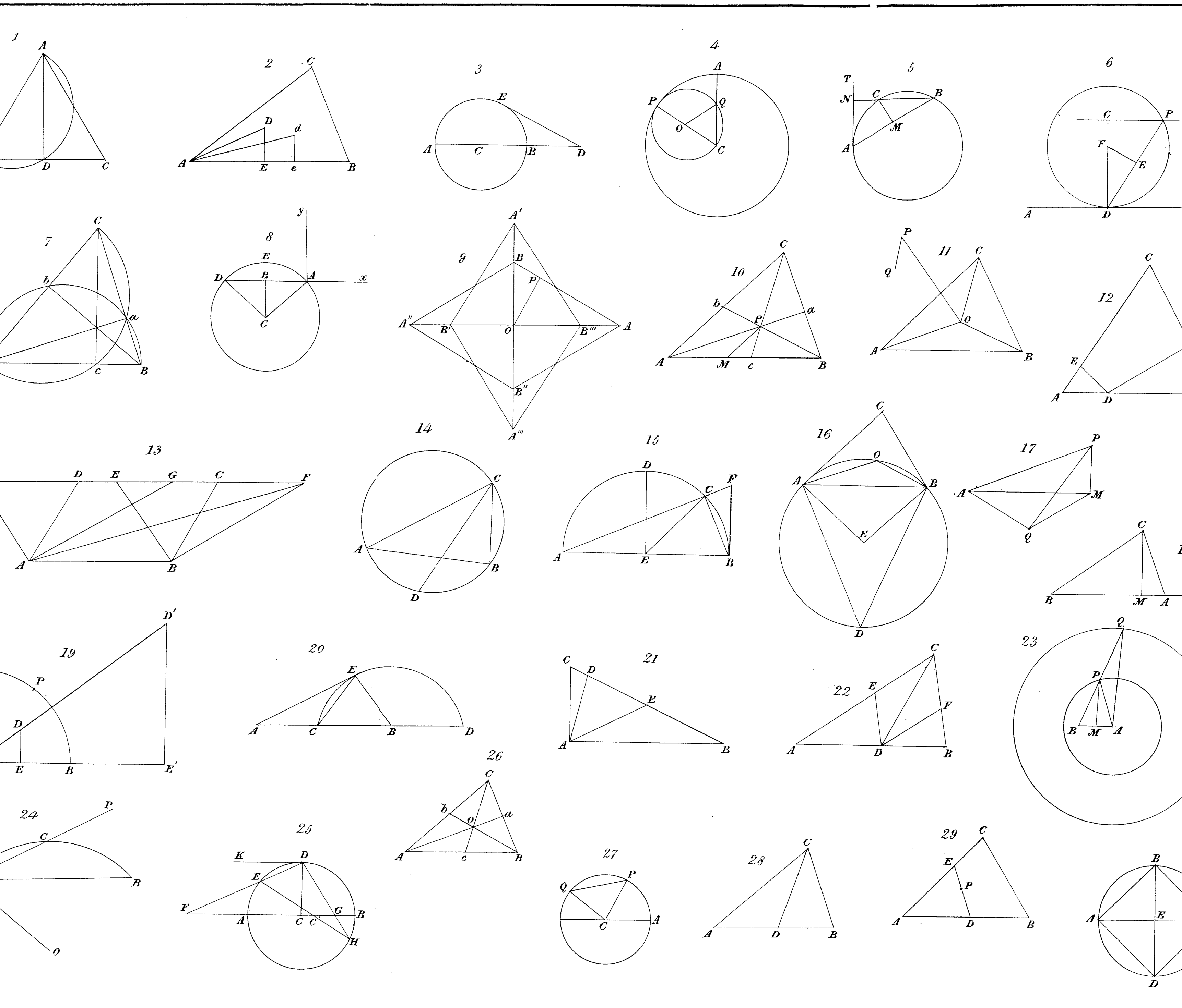

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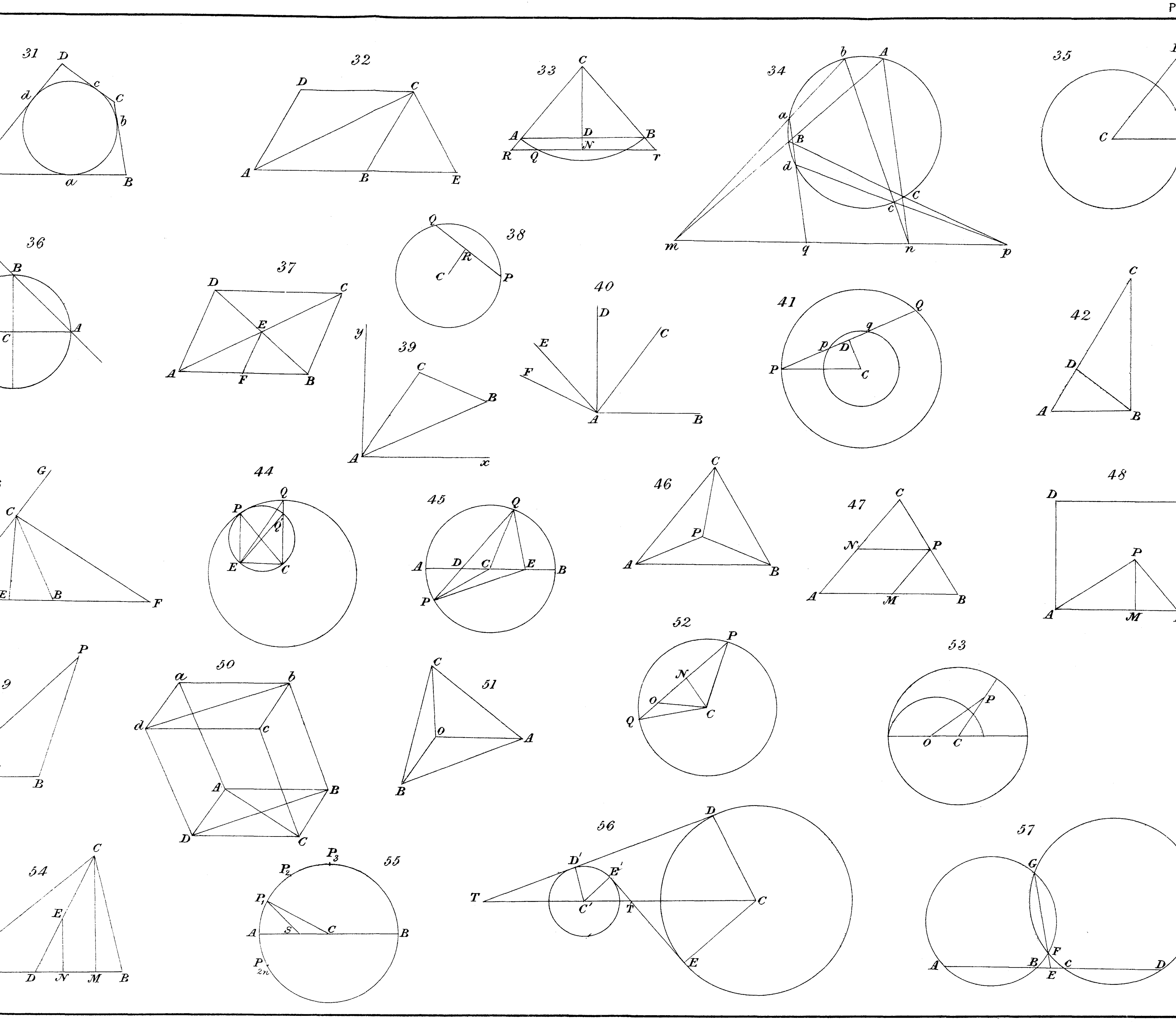

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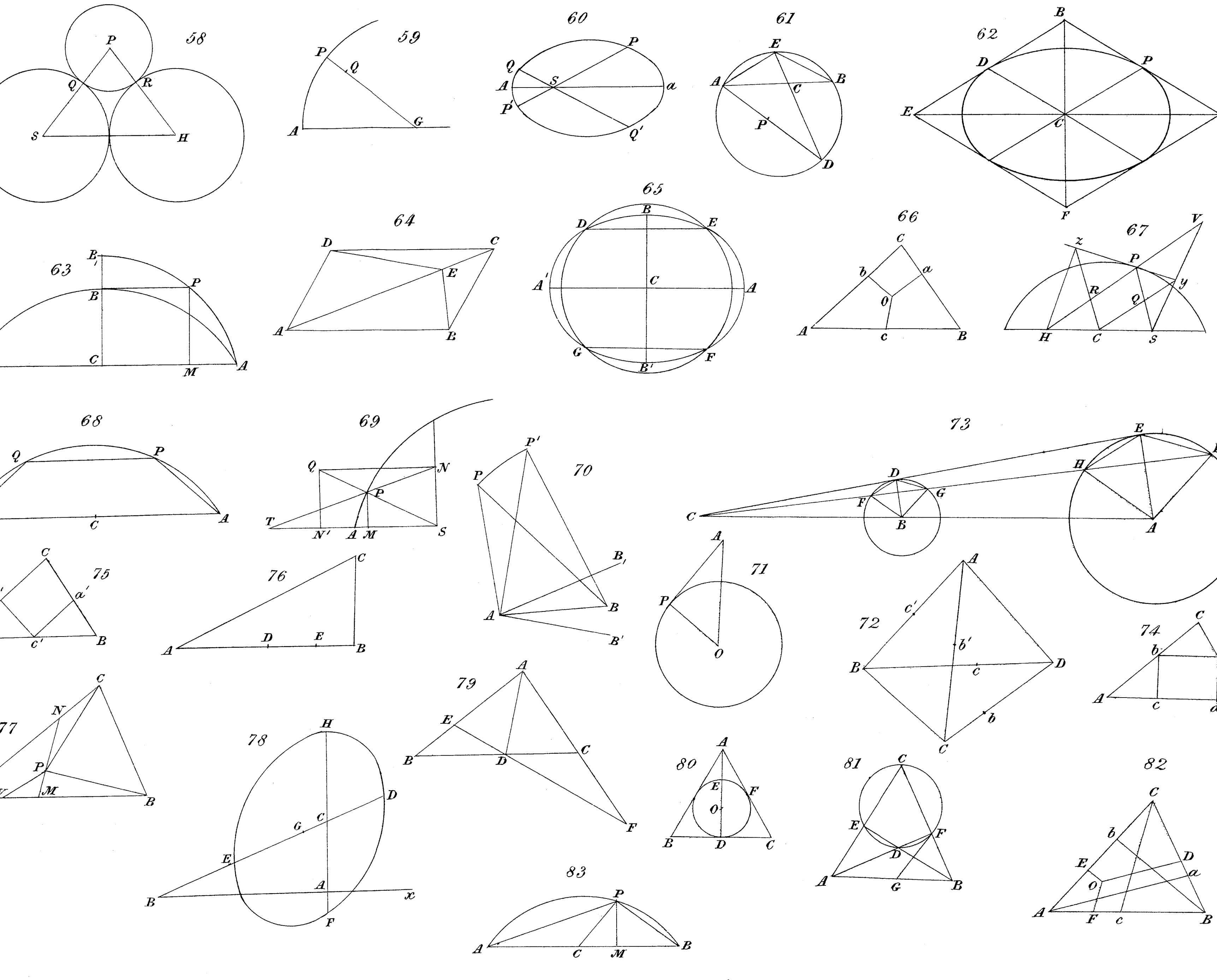

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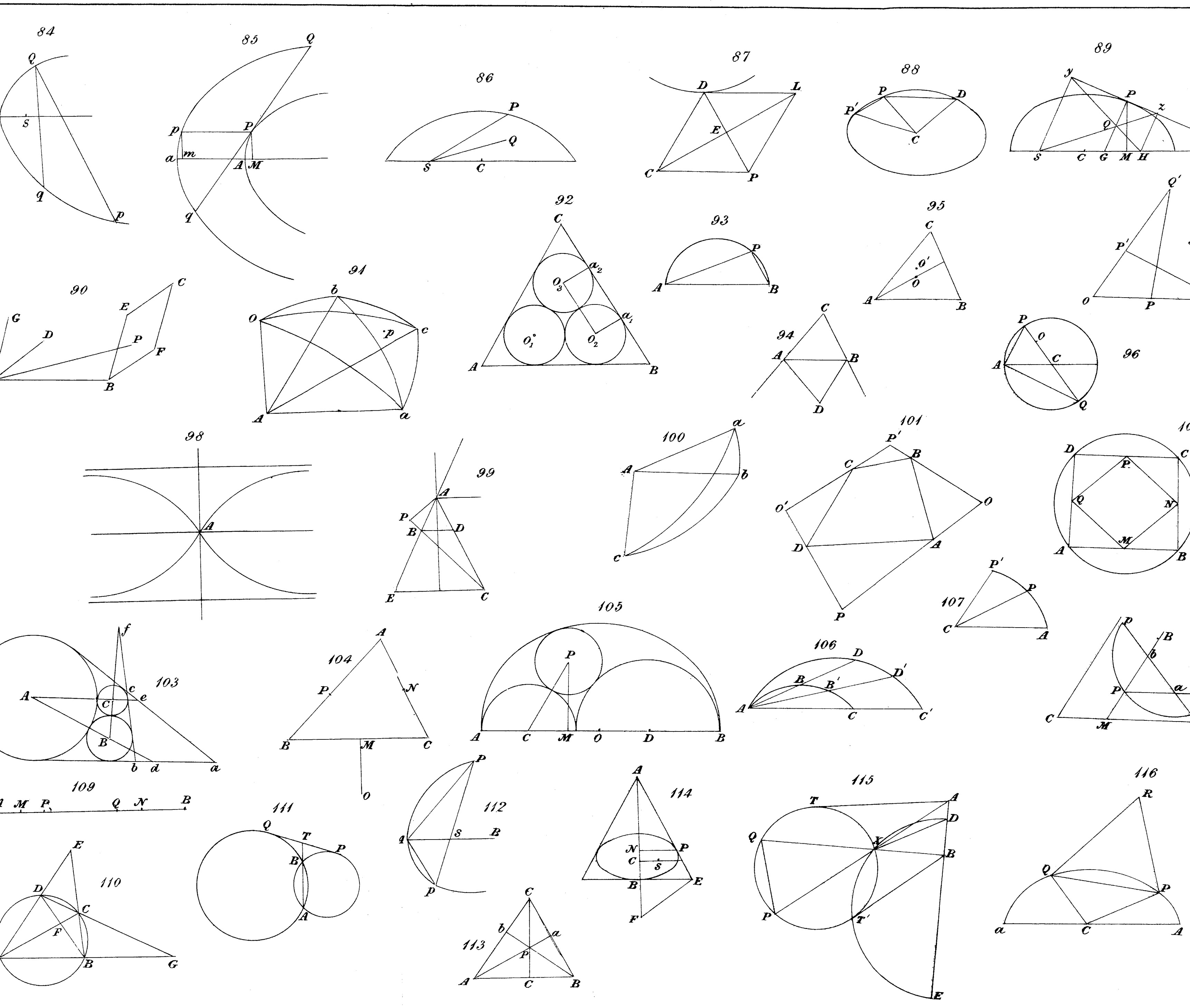

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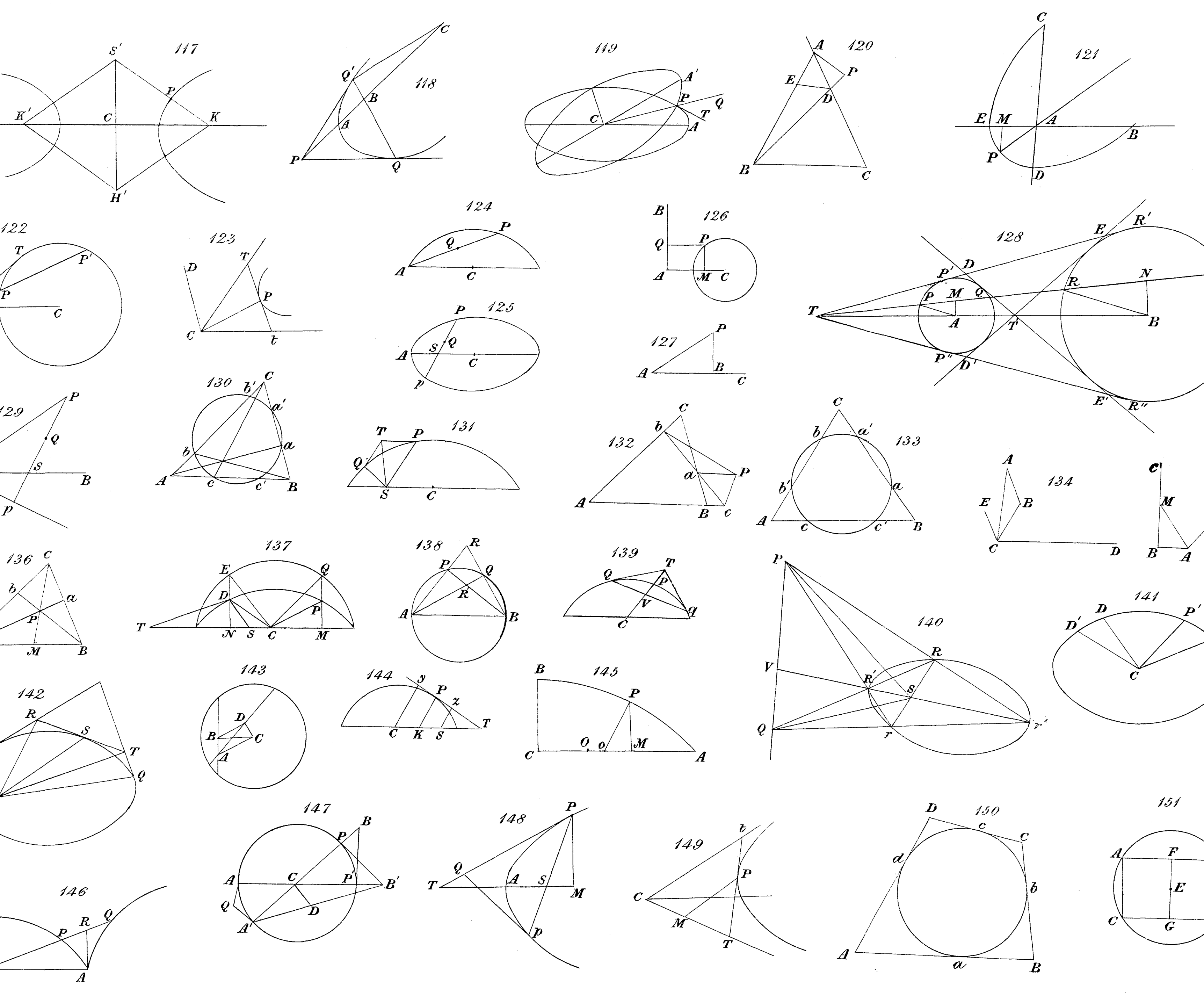

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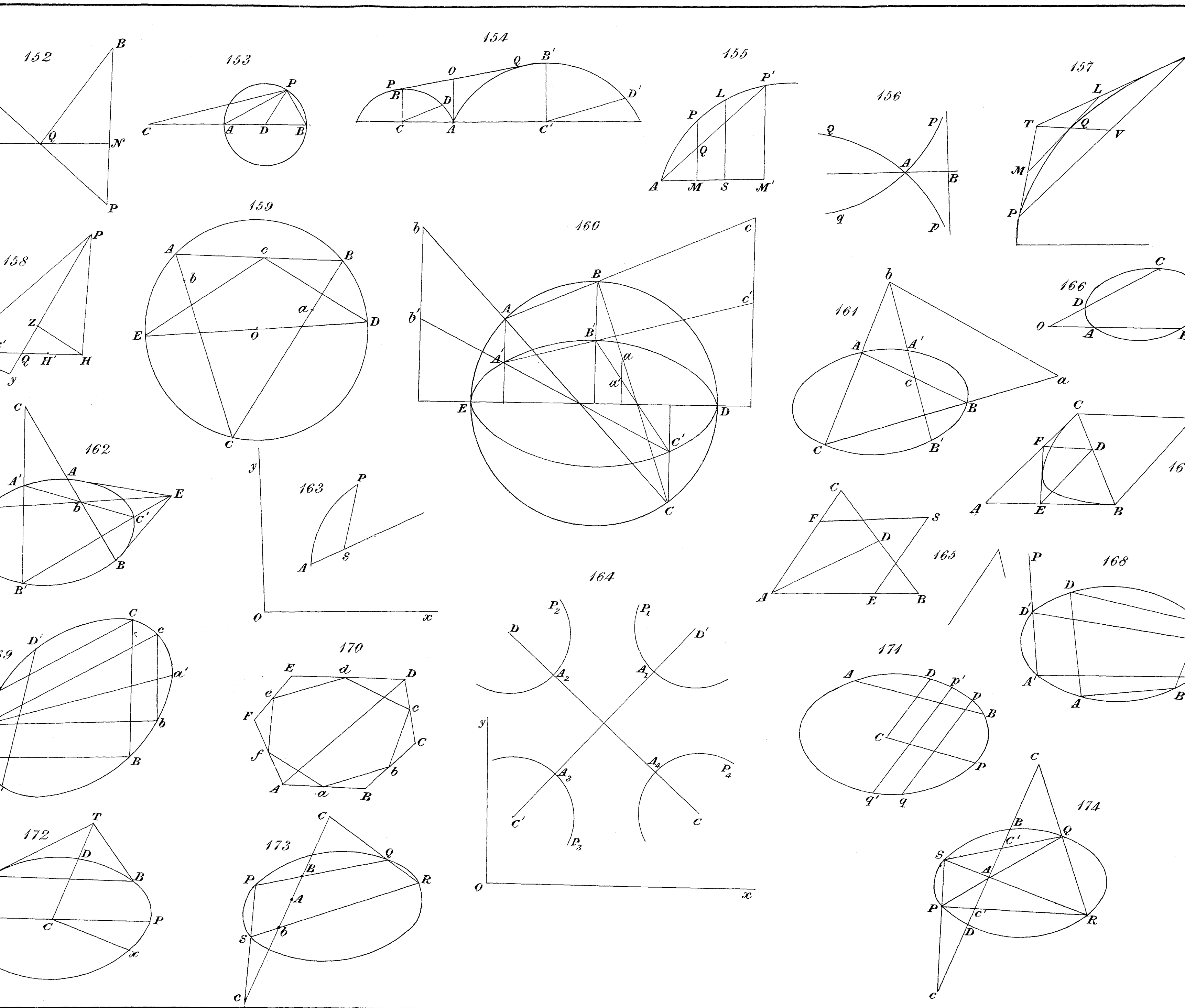

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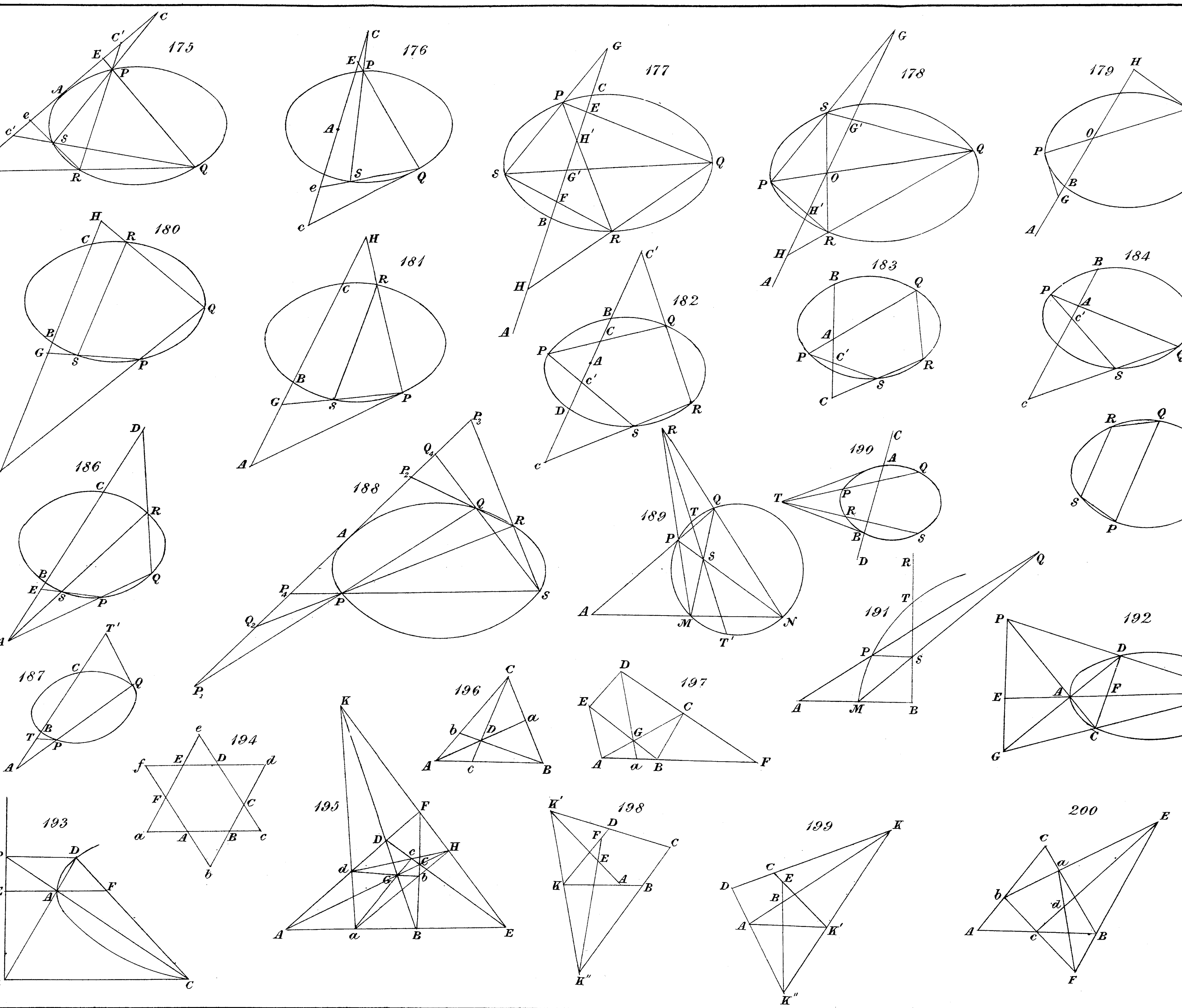
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